4

SPECIAL PRODUCTS AND FACTORIZATION

In an earlier lesson you have learnt multiplication of algebraic expressions, particularly polynomials. In the study of algebra, we come across certain products which occur very frequently. By becoming familiar with them, a lot of time and labour can be saved as in those products, multiplication is performed without actually writing down all the steps. For example, products, such as 108 × 108, 97 × 97, 104 × 96, 99 × 99 × 99, can be easily calculated if you know the products $(a + b)^2$, $(a - b)^2$, $(a + b)(a - b)$, $(a - b)^3$ respectively. Such products are called **special products**.

Factorization is a process of finding the factors of certain given products such as $a^2 - b^2$, $a^3 + 8b^3$, etc. We will consider factoring only those polynomials in which coefficients are integers.

In this lesson, you will learn about certain special products and factorization of certain polynomials. Besides, you will learn about finding HCF and LCM of polynomials by factorization. In the end you will be made familiar with rational algebraic expressions and to perform fundamental operations on rational expressions.

OBJECTIVES

After studying this lesson, you will be able to

- *write formulae for special products* $(a \pm b)^2$, $(a + b)(a - b)$, $(x + a)(x + b)$, $(a + b)(a^2 - ab + b^2)$, $(a - b)(a^2 + ab + b^2)$, $(a \pm b)^3$ *and* $(ax + b)(cx + d)$;
- *calculate squares and cubes of numbers using formulae;*
- *factorise given polynomials including expressions of the forms* $a^2 - b^2$, $a^3 \pm b^3$;
- *factorise polynomials of the form* $ax^2 + bx + c$ $(a \neq 0)$ *by splitting the middle term;*
- *determine HCF and LCM of polynomials by factorization;*

- *cite examples of rational expressions in one and two variables;*
- *perform four fundamental operations on rational expressions.*

EXPECTED BACKGROUND KNOWLEDGE

- Number system and four fundamental operations
- Laws of exponents
- Algebraic expressions
- Four fundamental operations on polynomials
- HCF and LCM of numbers
- Elementary concepts of geometry and mensuration learnt at primary and upper primary levels.

4.1 SPECIAL PRODUCTS

Here, we consider some speical products which occur very frequently in algebra.

(1) Let us find $(a + b)^2$

$(a + b)^2 = (a + b)(a + b)$

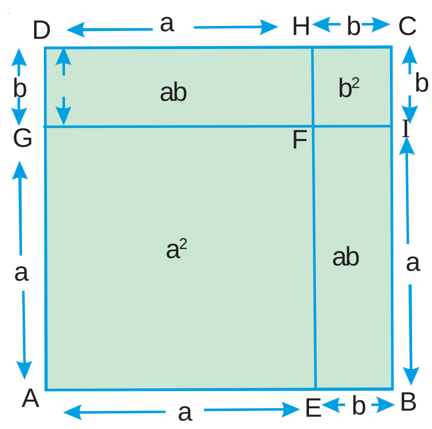
$= a(a + b) + b(a + b)$ [Distributive law]

$= a^2 + ab + ab + b^2$

$= a^2 + 2ab + b^2$

Geometrical verification

Concentrate on the figure, given here, on the right



(i) $(a + b)^2$ = Area of square ABCD

= Area of square AEFG + area of rectangle EBIF + area of rectangle DGFH + area of square CHFI

$= a^2 + ab + ab + b^2$

$= a^2 + 2ab + b^2$

Thus, $\mathbf{(a + b)^2 = a^2 + 2ab + b^2}$

(2) Let us find $(a - b)^2$

$(a - b)^2 \quad = (a - b)(a - b)$ [Distributive law]

$= a(a - b) - b(a - b)$

$= a^2 - ab - ab + b^2$

$= a^2 - 2ab + b^2$

Method 2: Using $(a + b)^2$

We know that $a - b = a + (-b)$

$\therefore (a - b)^2 = [a + (-b)]^2$

$= a^2 + 2(a)(-b) + (-b)^2$

$= a^2 - 2ab + b^2$

Geometrical verification

Concentrate on the figure, given here, on the right

$(a - b)^2$ = Area of square PQRS

= Area of square STVX –

[area of rectangle RTVW +

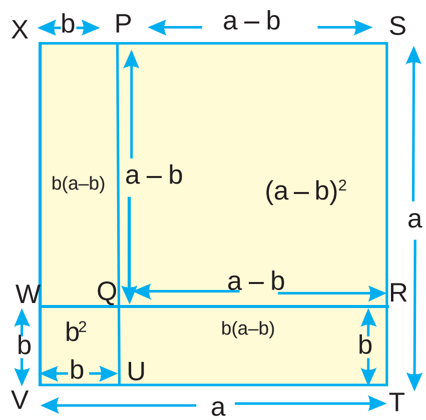
area of rectangle PUVX –

area of square QUVW]

$= a^2 - (ab + ab - b^2)$

$= a^2 - ab - ab + b^2$

$= a^2 - 2ab + b^2$



Thus, $\mathbf{(a - b)^2 = a^2 - 2ab + b^2}$

Deductions: We have

$(a + b)^2 = a^2 + 2ab + b^2$ .....(1)

$(a - b)^2 = a^2 - 2ab + b^2$ .....(2)

(1) + (2) gives

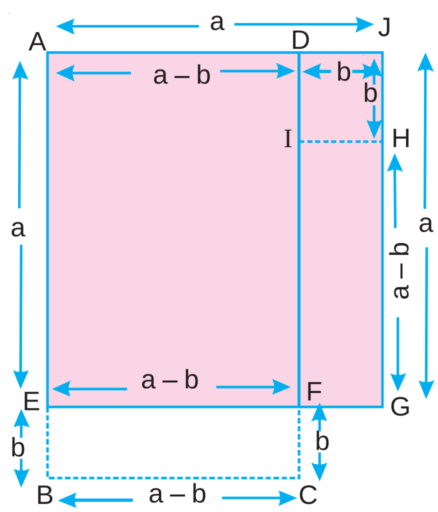
$\mathbf{(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)}$

(1) – (2) gives

$\mathbf{(a + b)^2 - (a - b)^2 = 4ab}$

(3) Now we find the product $(a + b)(a - b)$

$$(a + b)(a - b) = a(a - b) + b(a - b) \quad \text{[Distributive law]}$$
$$= a^2 - ab + ab - b^2$$
$$= a^2 - b^2$$



Geometrical verification

Observe the figure, given here, on the right

$(a + b)(a - b)$ = Area of Rectangle ABCD

= Area of Rectangle AEFD + area of rectangle EBCF

= Area of Rectangle AEFD + Area of Rectangle FGHI

= [Area of Rectangle AEFD + Area of rectangle FGHI + Area of square DIHJ] – Area of square DIHJ

= Area of square AEGJ – area of square DIHJ

$= a^2 - b^2$

Thus, $\mathbf{(a + b)(a - b) = a^2 - b^2}$

The process of multiplying the sum of two numbers by their difference is very useful in arithmetic. For example,

$$64 \times 56 = (60 + 4) \times (60 - 4)$$
$$= 60^2 - 4^2$$
$$= 3600 - 16$$
$$= 3584$$

(4) We, now find the product $(x + a)(x + b)$

$$(x + a)(x + b) = x(x + b) + a(x + b) \quad \text{[Distributive law]}$$
$$= x^2 + bx + ax + ab$$
$$= x^2 + (a + b)x + ab$$

Thus, $\mathbf{(x + a)(x + b) = x^2 + (a + b)x + ab}$

Deductions: (i) $\mathbf{(x - a)(x - b) = x^2 - (a + b)x + ab}$

(ii) $\mathbf{(x - a)(x + b) = x^2 + (b - a)x - ab}$

Students are advised to verify these results.

(5) Let us, now, find the product (ax + b) (cx + d)

$$(ax + b)(cx + d) = ax(cx + d) + b(cx + d)$$
$$= acx^2 + adx + bcx + bd$$
$$= acx^2 + (ad + bc)x + bd$$

Thus, $\mathbf{(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd}$

Deductions: (i) $\mathbf{(ax - b)(cx - d) = acx^2 - (ad + bc)x + bd}$

(ii) $\mathbf{(ax - b)(cx + d) = acx^2 - (bc - ad)x - bd}$

Students should verify these results.

Let us, now, consider some examples based on the special products mentioned above.

Example 4.1: Find the following products:

(i) $(2a + 3b)^2$ (ii) $\left(\frac{3}{2}a - 6b\right)^2$

(iii) $(3x + y)(3x - y)$ (iv) $(x + 9)(x + 3)$

(v) $(a + 15)(a - 7)$ (vi) $(5x - 8)(5x - 6)$

(vii) $(7x - 2a)(7x + 3a)$ (viii) $(2x + 5)(3x + 4)$

Solution:

(i) Here, in place of a, we have 2a and in place of b, we have 3b.

$$(2a + 3b)^2 = (2a)^2 + 2(2a)(3b) + (3b)^2$$
$$= 4a^2 + 12ab + 9b^2$$

(ii) Using special product (2), we get

$$\left(\frac{3}{2}a - 6b\right)^2 = \left(\frac{3}{2}a\right)^2 - 2\left(\frac{3}{2}a\right)(6b) + (6b)^2$$

$$= \frac{9}{4}a^2 - 18ab + 36b^2$$

(iii) $(3x + y)(3x - y) = (3x)^2 - y^2$ [using speical product (3)]

$$= 9x^2 - y^2$$

(iv) $(x + 9)(x + 3) = x^2 + (9 + 3)x + 9 \times 3$ [using speical product (4)]

$= x^2 + 12x + 27$

(v) $(a + 15)(a - 7) = a^2 + (15 - 7)a - 15 \times 7$

$= a^2 + 8a - 105$

(vi) $(5x - 8)(5x - 6) = (5x)^2 - (8 + 6)(5x) + 8 \times 6$

$= 25x^2 - 70x + 48$

(vii) $(7x - 2a)(7x + 3a) = (7x)^2 + (3a - 2a)(7x) - (3a)(2a)$

$= 49x^2 + 7ax - 6a^2$

(viii) $(2x + 5)(3x + 4) = (2 \times 3)x^2 + (2 \times 4 + 5 \times 3)x + 5 \times 4$

$= 6x^2 + 23x + 20$

Numerical calculations can be performed more conveniently with the help of special products, often called **algebraic formulae.** Let us consider the following example.

Example 4.2: Using special products, calculate each of the following:

(i) 101×101 (ii) 98×98 (iii) 68×72

(iv) 107×103 (v) 56×48 (vi) 94×99

Solution: (i) $101 \times 101 = 101^2 = (100 + 1)^2$

$= 100^2 + 2 \times 100 \times 1 + 1^2$

$= 10000 + 200 + 1$

$= 10201$

(ii) $98 \times 98 = 98^2 = (100 - 2)^2$

$= 100^2 - 2 \times 100 \times 2 + 2^2$

$= 10000 - 400 + 4$

$= 9604$

(iii) $68 \times 72 = (70 - 2) \times (70 + 2)$

$= 70^2 - 2^2$

$= 4900 - 4$

$= 4896$

(iv) $107 \times 103 = (100 + 7)(100 + 3)$

$= 100^2 + (7 + 3) \times 100 + 7 \times 3$

$= 10000 + 1000 + 21$

$= 11021$

(v) $56 \times 48 = (50+6)(50-2)$

$= 50^2 + (6-2) \times 50 - 6 \times 2$

$= 2500 + 200 - 12$

$= 2688$

(vi) $94 \times 99 = (100-6)(100-1)$

$= 100^2 - (6+1) \times 100 + 6 \times 1$

$= 10000 - 700 + 6$

$= 9306$

CHECK YOUR PROGRESS 4.1

1. Find each of the following products:

(i) $(5x+y)^2$ (ii) $(x-3)^2$ (iii) $(ab+cd)^2$

(iv) $(2x-5y)^2$ (v) $\left(\frac{x}{3}+1\right)^2$ (vi) $\left(\frac{z}{2}-\frac{1}{3}\right)^2$

(vii) $(a^2+5)(a^2-5)$ (viii) $(xy-1)(xy+1)$ (ix) $\left(x+\frac{4}{3}\right)\left(x+\frac{3}{4}\right)$

(x) $\left(\frac{2}{3}x^2-3\right)\left(\frac{2}{3}x^2+\frac{1}{3}\right)$ (xi) $(2x+3y)(3x+2y)$ (xii) $(7x+5y)(3x-y)$

2. Simplify:

(i) $(2x^2+5)^2-(2x^2-5)^2$ (ii) $(a^2+3)^2+(a^2-3)^2$

(iii) $(ax+by)^2+(ax-by)^2$ (iv) $(p^2+8q^2)^2-(p^2-8q^2)^2$

3. Using special products, calculate each of the following:

(i) 102×102 (ii) 108×108 (iii) 69×69

(iv) 998×998 (v) 84×76 (vi) 157×143

(vii) 306×294 (viii) 508×492 (ix) 105×109

(x) 77×73 (xi) 94×95 (xii) 993×996

4.2 SOME OTHER SPECIAL PRODUCTS

(6) Consider the binomial (a + b). Let us find its cube.

$(a + b)^3 = (a + b)(a + b)^2$

$= (a + b)(a^2 + 2ab + b^2)$ [using laws of exponents)

$= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)$ [Distributive laws]

$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$

$= a^3 + 3a^2b + 3ab^2 + b^3$

$= a^3 + 3ab(a + b) + b^3$

Thus, $\mathbf{(a + b)^3 = a^3 + 3ab(a + b) + b^3}$

(7) We now find the cube of (a – b).

$(a - b)^3 = (a - b)(a - b)^2$

$= (a - b)(a^2 - 2ab + b^2)$ [using laws of exponents)

$= a(a^2 - 2ab + b^2) - b(a^2 - 2ab + b^2)$ [Distributive laws]

$= a^3 - 2a^2b + ab^2 - a^2b + 2ab^2 - b^3$

$= a^3 - 3a^2b + 3ab^2 - b^3$

$= a^3 - 3ab(a - b) - b^3$

Thus, $\mathbf{(a - b)^3 = a^3 - 3ab(a - b) - b^3}$

Note: You may also get the same result on replacing b by –b in

$(a + b)^3 = a^3 + 3ab(a + b) + b^3$

(8) $(a + b)(a^2 - ab + b^2) = a(a^2 - ab + b^2) + b(a^2 - ab + b^2)$ [Distributive law]

$= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$

$= a^3 + b^3$

Thus, $\mathbf{(a + b)(a^2 - ab + b^2) = a^3 + b^3}$

(9) $(a - b)(a^2 + ab + b^2) = a(a^2 + ab + b^2) - b(a^2 + ab + b^2)$ [Distributive law]

$= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3$

$= a^3 - b^3$

Thus, $\mathbf{(a - b)(a^2 + ab + b^2) = a^3 - b^3}$

Let us, now, consider some examples based on the above mentioned special products:

Example 4.3: Find each of the following products:

(i) $(7x + 9y)^3$ (ii) $(px - yz)^3$ (iii) $(x - 4y^2)^3$

(iv) $(2a^2 + 3b^2)^3$ (v) $\left(\frac{2}{3}a - \frac{5}{3}b\right)^3$ (vi) $\left(1 + \frac{4}{3}c\right)^3$

Solution: (i) $(7x + 9y)^3 = (7x)^3 + 3(7x)(9y)(7x + 9y) + (9y)^3$

$= 343x^3 + 189xy(7x + 9y) + 729y^3$

$= 343x^3 + 1323x^2y + 1701xy^2 + 729y^3$

(ii) $(px - yz)^3 = (px)^3 - 3(px)(yz)(px - yz) - (yz)^3$

$= p^3x^3 - 3pxyz(px - yz) - y^3z^3$

$= p^3x^3 - 3p^2x^2yz + 3pxy^2z^2 - y^3z^3$

(iii) $(x - 4y^2)^3 = x^3 - 3x(4y^2)(x - 4y^2) - (4y^2)^3$

$= x^3 - 12xy^2(x - 4y^2) - 64y^6$

$= x^3 - 12x^2y^2 + 48xy^4 - 64y^6$

(iv) $(2a^2 + 3b^2)^3 = (2a^2)^3 + 3(2a^2)(3b^2)(2a^2 + 3b^2) + (3b^2)^3$

$= 8a^6 + 18a^2b^2(2a^2 + 3b^2) + 27b^6$

$= 8a^6 + 36a^4b^2 + 54a^2b^4 + 27b^6$

(v) $$\left(\frac{2}{3}a - \frac{5}{3}b\right)^3 = \left(\frac{2}{3}a\right)^3 - 3\left(\frac{2}{3}a\right)\left(\frac{5}{3}b\right)\left(\frac{2}{3}a - \frac{5}{3}b\right) - \left(\frac{5}{3}b\right)^3$$

$$= \frac{8}{27}a^3 - \frac{10}{3}ab\left(\frac{2}{3}a - \frac{5}{3}b\right) - \frac{125}{27}b^3$$

$$= \frac{8}{27}a^3 - \frac{20}{9}a^2b + \frac{50}{9}ab^2 - \frac{125}{27}b^3$$

(vi) $$\left(1 + \frac{4}{3}c\right)^3 = (1)^3 + 3(1)\left(\frac{4}{3}c\right)\left(1 + \frac{4}{3}c\right) + \left(\frac{4}{3}c\right)^3$$

$$= 1 + 4c\left(1 + \frac{4}{3}c\right) + \frac{64}{27}c^3$$

$$= 1 + 4c + \frac{16}{3}c^2 + \frac{64}{27}c^3$$

Example 4.4: Using special products, find the cube of each of the following:

(i) 19 (ii) 101 (iii) 54 (iv) 47

Solution: (i) $19^3 = (20-1)^3$

$= 20^3 - 3 \times 20 \times 1\,(20-1) - 1^3$

$= 8000 - 60\,(20-1) - 1$

$= 8000 - 1200 + 60 - 1$

$= 6859$

(ii) $101^3 = (100+1)^3$

$= 100^3 + 3 \times 100 \times 1\,(100+1) + 1^3$

$= 1000000 + 300 \times 100 + 300 + 1$

$= 1030301$

(iii) $54^3 = (50+4)^3$

$= 50^3 + 3 \times 50 \times 4\,(50+4) + 4^3$

$= 125000 + 600\,(50+4) + 64$

$= 125000 + 30000 + 2400 + 64$

$= 157464$

(iv) $47^3 = (50-3)^3$

$= 50^3 - 3 \times 50 \times 3\,(50-3) - 3^3$

$= 125000 - 450\,(50-3) - 27$

$= 125000 - 22500 + 1350 - 27$

$= 103823$

Example 4.5: Without actual multiplication, find each of the following products:

(i) $(2a+3b)(4a^2 - 6ab + 9b^2)$

(ii) $(3a-2b)(9a^2 + 6ab + 4b^2)$

Solution: (i) $(2a+3b)(4a^2 - 6ab + 9b^2) = (2a+3b)[(2a)^2 - (2a)(3b) + (3b)^2]$

$= (2a)^3 + (3b)^3$

$= 8a^3 + 27b^3$

(ii) $(3a-2b)(9a^2 + 6ab + 4b^2) = (3a-2b)[(3a)^2 + (3a)(2b) + (2b)^2]$

$= (3a)^3 - (2b)^3$

$= 27a^3 - 8b^3$

Example 4.6: Simplify:

(i) $(3x - 2y)^3 + 3(3x - 2y)^2(3x + 2y) + 3(3x - 2y)(3x + 2y)^2 + (3x + 2y)^3$

(ii) $(2a - b)^3 + 3(2a - b)(2b - a)(a + b) + (2b - a)^3$

Solution: (i) Put $3x - 2y = a$ and $3x + 2y = b$

The given expression becomes

$a^3 + 3a^2b + 3ab^2 + b^3$

$= (a + b)^3$

$= (3x - 2y + 3x + 2y)^3$

$= (6x)^3$

$= 216x^3$

(ii) Put $2a - b = x$ and $2b - a = y$ so that $a + b = x + y$

The given expression becomes

$x^3 + 3xy(x + y) + y^3$

$= (x + y)^3$

$= (a + b)^3$

$= a^3 + 3a^2b + 3ab^2 + b^3$

Example 4.7: Simplify:

(i) $\dfrac{857 \times 857 \times 857 - 537 \times 537 \times 537}{857 \times 857 + 857 \times 537 + 537 \times 537}$

(ii) $\dfrac{674 \times 674 \times 674 + 326 \times 326 \times 326}{674 \times 674 - 674 \times 326 + 326 \times 326}$

Solution: The given expression can be written as

$$\frac{857^3 - 537^3}{857^2 + 857 \times 537 + 537^2}$$

Let $857 = a$ and $537 = b$, then the expression becomes

$$\frac{a^3 - b^3}{a^2 + ab + b^2} = \frac{(a - b)(a^2 + ab + b^2)}{a^2 + ab + b^2} = a - b$$

$= 857 - 537$

$= 320$

(ii) The given expression can be written as

$$\frac{674^3 + 326^3}{674^2 - 674 \times 326 + 326^2}$$

$$= \frac{(674 + 326)(674^2 - 674 \times 326 + 326^2)}{674^2 - 674 \times 326 + 326^2}$$

$= 674 + 326$

$= 1000$

CHECK YOUR PROGRESS 4.2

1. Write the expansion of each of the following:

(i) $(3x + 4y)^3$ (ii) $(p - qr)^3$ (iii) $\left(a + \frac{b}{3}\right)^3$

(iv) $\left(\frac{a}{3} - b\right)^3$ (v) $\left(\frac{1}{2}a^2 + \frac{2}{3}b^2\right)^3$ (vi) $\left(\frac{1}{3}a^2x^3 - 2b^3y^2\right)^3$

2. Using special products, find the cube of each of the following:

(i) 8 (ii) 12 (iii) 18 (iv) 23

(v) 53 (vi) 48 (vii) 71 (viii) 69

(ix) 97 (x) 99

3. Without actual multiplication, find each of the following products:

(i) $(2x + y)(4x^2 - 2xy + y^2)$ (ii) $(x - 2)(x^2 + 2x + 4)$

(iii) $(1 + x)((1 - x + x^2)$ (iv) $(2y - 3z^2)(4y^2 + 6yz^2 + 9z^4)$

(v) $(4x + 3y)(16x^2 - 12xy + 9y^2)$ (vi) $\left(3x - \frac{1}{7}y\right)\left(9x^2 + \frac{3}{7}xy + \frac{1}{49}y^2\right)$

4. Find the value of:

(i) $a^3 + 8b^3$ if $a + 2b = 10$ and $ab = 15$

[Hint: $(a + 2b)^3 = a^3 + 8b^3 + 6ab(a + 2b) \Rightarrow a^3 + 8b^3 = (a + 2b)^3 - 6ab(a + 2b)$]

(ii) $x^3 - y^3$ when $x - y = 5$ and $xy = 66$

5. Find the value of $64x^3 - 125z^3$ if
   (i) $4x - 5z = 16$ and $xz = 12$
   (ii) $4x - 5z = \frac{3}{5}$ and $xz = 6$
6. Simplify:
   (i) $(2x + 5)^3 - (2x - 5)^3$
   (ii) $(7x + 5y)^3 - (7x - 5y)^3 - 30y\,(7x + 5y)\,(7x - 5y)^3$
   [Hint put $7x + 5y = a$ and $7x - 5y = b$ so that $a - b = 10y$]
   (iii) $(3x + 2y)\,(9x^2 - 6xy + 4y^2) - (2x + 3y)\,(4x^2 - 6xy + 9y^2)$
   (iv) $(2x - 5)\,(4x^2 + 10x + 25) - (5x + 1)\,(25x^2 - 5x + 1)$
7. Simplify:
   (i) $\dfrac{875 \times 875 \times 875 + 125 \times 125 \times 125}{875 \times 875 - 875 \times 125 + 125 \times 125}$
   (ii) $\dfrac{678 \times 678 \times 678 - 234 \times 234 \times 234}{678 \times 678 + 678 \times 234 + 234 \times 234}$

4.3 FACTORIZATION OF POLYNOMIALS

Recall that from $3 \times 4 = 12$, we say that 3 and 4 are factors of the product 12. Similarly, in algebra, since $(x + y)\,(x - y) = x^2 - y^2$, we say that $(x + y)$ and $(x - y)$ are factors of the product $(x^2 - y^2)$.

Factorization of a polynomial is a process of writing the polynomial as a product of two (or more) polynomials. Each polynomial in the product is called a factor of the given polynomial.

In factorization, we shall restirct ourselves, unless otherwise stated, to finding factors of the polynomials over integers, i.e. polynomials with integral coefficients. In such cases, it is required that the factors, too, be polynomials over integers. Polynomials of the type $2x^2 - y^2$ will not be considered as being factorable into $\left(\sqrt{2}x + y\right)\left(\sqrt{2}x - y\right)$ because these factors are not polynomials over integers.

A polynomial will be said to be completely factored if none of its factors can be further expressed as a product of two polynomials of lower degree and if the integer coefficients have no common factor other than 1 or –1. Thus, complete factorization of $(x^2 - 4x)$ is $x(x-4)$. On the other hand the factorization $(4x^2 - 1)\,(4x^2 + 1)$ of $(16x^4 - 1)$ is not complete since the factor $(4x^2 - 1)$ can be further factorised as $(2x - 1)\,(2x + 1)$. Thus, complete factorization of $(16x^4 - 1)$ is $(2x - 1)\,(2x + 1)\,(4x^2 + 1)$.

In factorization, we shall be making full use of special products learnt earlier in this lesson. Now, in factorization of polynomials we take various cases separately through examples.

(1) Factorization by Distributive Property

Example 4.8: Factorise:

(i) $10a - 25$ (ii) $x^2y^3 + x^3y^2$

(iii) $5ab(ax^2 + y^2) - 6mn(ax^2 + y^2)$ (iv) $a(b - c)^2 + b(b - c)$

Solution: (i) $10a - 25 = 5 \times 2a - 5 \times 5$

$= 5(2a - 5)$ [Since 5 is common to the two terms]

Thus, 5 and $2a - 5$ are factors of $10a - 25$

(ii) In $x^2y^3 + x^3y^2$, note that x^2y^2 is common (with greatest degree) in both the terms.

$$\therefore x^2y^3 + x^3y^2 = x^2y^2 \times y + x^2y^2 \times x$$

$$= x^2y^2(y + x)$$

Therefore, $x, x^2, y, y^2, xy, x^2y, xy^2, x^2y^2$ and $y + x$ are factors of $x^2y^3 + x^3y^2$

(iii) Note that $ax^2 + y^2$ is common in both the terms

$$\therefore 5ab(ax^2 + y^2) - 6mn(ax^2 + y^2) = (ax^2 + y^2)(5ab - 6mn)$$

(iv) $a(b - c)^2 + b(b - c) = (b - c) \times [a(b - c)] + (b - c) \times b$

$= (b - c) \times [a(b - c) + b]$

$= (b - c) \times [ab - ac + b]$

(2) Factorization Involving the Difference of Two Squares

You know that $(x + y)(x - y) = x^2 - y^2$. Therefore $x + y$ and $x - y$ are factors of $x^2 - y^2$.

Example 4.9: Factorise:

(i) $9x^2 - 16y^2$ (ii) $x^4 - 81y^4$

(iii) $a^4 - (2b - 3c)^2$ (iv) $x^2 - y^2 + 6y - 9$

Solution: (i) $9x^2 - 16y^2 = (3x)^2 - (4y)^2$ which is a difference of two squares.

$= (3x + 4y)(3x - 4y)$

(ii) $x^4 - 81y^4 = (x^2)^2 - (9y^2)$

$= (x^2 + 9y^2)(x^2 - 9y^2)$

Note that $x^2 - 9y^2 = (x)^2 - (3y)^2$ is again a difference of the two squares.

$x^4 - 81y^4 = (x^2 + 9y^2)[(x)^2 - (3y)^2]$

$= (x^2 + 9y^2)(x + 3y)(x - 3y)$

(iii) $a^4 - (2b - 3c)^2 = (a^2)^2 - (2b - 3c)^2$

$= [a^2 + (2b - 3c)]\,[a^2 - (2b - 3c)]$

$= (a^2 + 2b - 3c)\,(a^2 - 2b + 3c)$

(iv) $x^2 - y^2 + 6y - 9 \quad = x^2 - (y^2 - 6y + 9)$ [Note this step)

$= (x)^2 - [(y)^2 - 2 \times y \times 3 + (3)^2]$

$= (x)^2 - (y - 3)^2$

$= [x + (y - 3)]\,[x - (y - 3)]$

$= (x + y - 3)\,(x - y + 3)$

(3) Factorization of a Perfect Square Trinomial

Example 4.10 : Factorise

(i) $9x^2 + 24xy + 16y^2$ (ii) $x^6 - 8x^3 + 16$

Solution: (i) $9x^2 + 24xy + 16y^2 \quad = (3x)^2 + 2\,(3x)\,(4y) + (4y)^2$

$= (3x + 4y)^2$

$= (3x + 4y)\,(3x + 4y)$

Thus, the two factors of the given polynomial are identical, each being $(3x + 4y)$.

(ii) $x^6 - 8x^3 + 16 \quad = (x^3)^2 - 2(x^3)\,(4) + (4)^2$

$= (x^3 - 4)^2$

$= (x^3 - 4)\,(x^3 - 4)$

Again, the two factors of the given polynomial are identical, each being $(x^3 - 4)$.

(4) Factorization of a Polynomial Reducible to the Difference of Two Squares

Example 4.11: Factorise

(i) $x^4 + 4y^4$ (ii) $x^4 + x^2 + 1$

Solution: (i) $x^4 + 4y^4 \quad = (x^2)^2 + (2y^2)^2$

$= (x^2)^2 + (2y^2)^2 + 2\,(x^2)\,(2y^2) - 2\,(x^2)\,(2y^2)$

[Adding and subtracting $2\,(x^2)\,(2y^2)$]

$= (x^2 + 2y^2)^2 - (2xy)^2$

$= (x^2 + 2y^2 + 2xy)\,(x^2 + 2y^2 - 2xy)$

(ii) $x^4 + x^2 + 1 = (x^2)^2 + (1)^2 + 2x^2 - x^2$

[Adding and subtracting x^2]

$= (x^2 + 1)^2 - (x)^2$

$= (x^2 + x + 1)(x^2 - x + 1)$

CHECK YOUR PROGRESS 4.3

Factorise:

1. $10xy - 15xz$
2. $abc^2 - ab^2c$
3. $6p^2 - 15pq + 27p$
4. $a^2(b - c) + b(c - b)$
5. $2a(4x - y)^3 - b(4x - y)^2$
6. $x(x + y)^3 - 3xy(x + y)$
7. $100 - 25p^2$
8. $1 - 256y^8$
9. $(2x + 1)^2 - 9x^2$
10. $(a^2 + bc)^2 - a^2(b + c)^2$
11. $25x^2 - 10x + 1 - 36y^2$
12. $49x^2 - 1 - 14xy + y^2$
13. $m^2 + 14m + 49$
14. $4x^2 - 4x + 1$
15. $36a^2 + 25 + 60a$
16. $x^6 - 8x^3 + 16$
17. $a^8 - 47a^4 + 1$
18. $4a^4 + 81b^4$
19. $x^4 + 4$
20. $9a^4 - a^2 + 16$
21. Find the value of n if

    (i) $6n = 23 \times 23 - 17 \times 17$ (ii) $536 \times 536 - 36 \times 36 = 5n$

(5) Factorization of Perfect Cube Polynomials

Example 4.12: Factorise:

(i) $x^3 + 6x^2y + 12xy^2 + 8y^3$ (ii) $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$

Solution: (i) $x^3 + 6x^2y + 12xy^2 + 8y^3$

$= (x)^3 + 3x^2(2y) + 3x(2y)^2 + (2y)^3$

$= (x + 2y)^3$

Thus, the three factors of the given polynomial are identical, each being $x + 2y$.

(ii) Given polynomial is equal to

$(x^2)^3 - 3x^2y^2(x^2 - y^2) - (y^2)^3$

$= (x^2 - y^2)^3$

$= [(x + y)(x - y)]^3$ [Since $x^2 - y^2 = (x + y)(x - y)$]

$= (x + y)^3 (x - y)^3$

(6) Factorization of Polynomials Involving Sum or Difference of Two Cubes

In special products you have learnt that

$$(x + y)(x^2 - xy + y^2) = x^3 + y^3$$

and $$(x - y)(x^2 + xy + y^2) = x^3 - y^3$$

Therefore, the factors of $x^3 + y^3$ are $x + y$ and $x^2 - xy + y^2$ and

those of $x^3 - y^3$ are $x - y$ and $x^2 + xy + y^2$

Now, consider the following example:

Example 4.13: Factorise

(i) $64a^3 + 27b^3$ (ii) $8x^3 - 125y^3$

(iii) $8(x + 2y)^3 - 343$ (iv) $a^4 - a^{13}$

Solution: (i) $64a^3 + 27b^3 = (4a)^3 + (3b)^3$

$= (4a + 3b)[(4a)^2 - (4a)(3b) + (3b)^2]$

$= (4a + 3b)(16a^2 - 12ab + 9b^2)$

(ii) $8x^3 - 125y^3 = (2x)^3 - (5y)^3$

$= (2x - 5y)[(2x)^2 + (2x)(5y) + (5y)^2]$

$= (2x - 5y)(4x^2 + 10xy + 25y^2)$

(iii) $8(x + 2y)^3 - 343 = [2(x + 2y)]^3 - (7)^3$

$= [2(x + 2y) - 7][2^2(x + 2y)^2 + 2(x + 2y)(7) + 7^2]$

$= (2x + 4y - 7)(4x^2 + 16xy + 16y^2 + 14x + 28y + 49)$

(iv) $a^4 - a^{13} = a^4(1 - a^9)$ [Since a^4 is common to the two terms]

$= a^4[(1)^3 - (a^3)^3]$

$= a^4(1 - a^3)(1 + a^3 + a^6)$

$= a^4(1 - a)(1 + a + a^2)(1 + a^3 + a^6)$

[Since $1 - a^3 = (1 - a)(1 + a + a^2)$]

CHECK YOUR PROGRESS 4.4

Factorise:

1. $a^3 + 216b^3$
2. $a^3 - 343$
3. $x^3 + 12x^2y + 48xy^2 + 64y^3$
4. $8x^3 - 36x^2y + 54xy^2 - 27y^3$

5. $8x^3 - 125y^3 - 60x^2y + 150xy^2$
6. $64k^3 - 144k^2 + 108k - 27$
7. $729\,x^6 - 8$
8. $x^2 + x^2\,y^6$
9. $16a^7 - 54ab^6$
10. $27b^3 - a^3 - 3a^2 - 3a - 1$
11. $(2a - 3b)^3 + 64c^3$
12. $64x^3 - (2y - 1)^3$

(7) Factorising Trinomials by Splitting the Middle Term

You have learnt that

$(x + a)(x + b) \quad = x^2 + (a + b)x + ab \quad = 1.x^2 + (a + b)x + ab$

and $(ax + b)(cx + d) \quad = acx^2 + (ad + bc)x + bd$

In general, the expressions given here on the right are of the form $Ax^2 + Bx + C$ which can be factorised by multiplying the coefficient of x^2 in the first term with the last term and finding two such factors of this product that their sum is equal to the coefficient of x in the second (middle) term. In other words, we are to determine two such factors of AC so that their sum is equal to B. The example, given below, will clarify the process further.

Example 4.14: Factorise:

(i) $x^2 + 3x + 2$ (ii) $x^2 - 10xy + 24y^2$

(iii) $5x^2 + 13x - 6$ (iv) $3x^2 - x - 2$

Solution: (i) Here, A = 1, B = 3 and C= 2; so $AC = 1 \times 2 = 2$

Therefore we are to determine two factors of 2 whose sum is 3

Obviously, $1 + 2 = 3$

(i.e. two factors of AC i.e. 2 are 1 and 2)

$\therefore$ We write the polynomial as

$x^2 + (1 + 2)\,x + 2$

$= x^2 + x + 2x + 2$

$= x(x + 1) + 2(x + 1)$

$= (x + 1)(x + 2)$

(ii) Here, $AC = 24y^2$ and $B = -10y$

Two factors of $24y^2$ whose sum is $-10y$ are $-4y$ and $-6y$

$\therefore$ We write the given polynomial as

$x^2 - 4xy - 6xy + 24\,y^2$

$= x(x - 4y) - 6y(x - 4y)$

$= (x - 4y)(x - 6y)$

(iii) Here, AC = $5 \times (-6) = -30$ and B = 13

Two factors of –30 whose sum is 13 are 15 and –2

$\therefore$ We write the given polynomial as

$5x^2 + 15x - 2x - 6$

$= 5x(x + 3) - 2(x + 3)$

$= (x + 3)(5x - 2)$

(iv) Here, AC = $3 \times (-2) = -6$ and B = – 1

Two factors of – 6 whose sum is (–1) are (–3) and 2.

$\therefore$ We write the given polynomial as

$3x^2 - 3x + 2x - 2$

$= 3x(x - 1) + 2(x - 1)$

$= (x - 1)(3x + 2)$

CHECK YOUR PROGRESS 4.5

Factorise:

1. $x^2 + 11x + 24$
2. $x^2 - 15xy + 54y^2$
3. $2x^2 + 5x - 3$
4. $6x^2 - 10xy - 4y^2$
5. $2x^4 - x^2 - 1$
6. $x^2 + 13xy - 30y^2$
7. $2x^2 + 11x + 14$
8. $10y^2 + 11y - 6$
9. $2x^2 - x - 1$
10. $(m - 1)(1 - m) + m + 109$
11. $(2a - b)^2 - (2a - b) - 30$
Hint put $2a - b = x$
12. $(2x + 3y)^2 - 2(2x + 3y)(3x - 2y) - 3(3x - 2y)^2$
Hint: Put $2x + 3y = a$ and $3x - 2y = b$

4.4 HCF AND LCM OF POLYNOMIALS

(1) HCF of Polynomials

You are already familiar with the term HCF (Highest Common Factor) of natural numbers in arithmetic. It is the largest number which is a factor of each of the given numbers. For instance, the HCF of 8 and 12 is 4 since the common factors of 8 and 12 are 1, 2 and 4 and 4 is the largest i.e. highest among them.

On similar lines in algebra, *the Highest Common Factor (HCF) of two or more given* *polynomials is the product of the polynomial(s) of highest degree and greatest numerical coefficient each of which is a factor of each of the given polynomials.*

For example, the HCF of $4(x + 1)^2$ and $6(x + 1)^3$ is $2(x + 1)^2$.

The HCF of monomials is found by multiplying the HCF of numerical coefficients of each of the monomials and the variable(s) with highest power(s) common to all the monomials. For example, the HCF of monomials $12x^2y^3$, $18xy^4$ and $24x^3y^5$ is $6xy^3$ since HCF of 12, 18 and 24 is 6; and the highest powers of variable factors common to the polynomials are x and y^3.

Let us now consider some examples.

Example 4.15: Find the HCF of

(i) $4x^2y$ and x^3y^2 (ii) $(x – 2)^3 (2x – 3)$ and $(x – 2)^2 (2x – 3)^3$

Solution: (i) HCF of numerical coefficients 4 and 1 is 1.

Since x occurs as a factor at least twice and y at least once in the given polynomials, therefore, their HCF is

$1 \times x^2 \times y$ i.e. x^2y

(ii) HCF of numerical coefficients 1 and 1 is 1.

In the given polynomials, $(x – 2)$ occurs as a factor at least twice and $(2x – 3)$ at least once. So the HCF of the given polynomials is

$1 \times (x – 2)^2 \times (2x – 3)$ i.e. $(x – 2)^2 (2x – 3)$

In view of Example 4.15 (ii), we can say that to determine the HCF of polynomials, which can be easily factorised, we express each of the polynomials as the product of the factors. Then the HCF of the given polynomials is the product of the HCF of numerical coefficients of each of the polynomials and factor (s) with highest power(s) common to all the polynomials. For further clarification, concentrate on the Example 4.16 given below.

Example 4.16: Find the HCF of

(i) $x^2 – 4$ and $x^2 + 4x + 4$

(ii) $4x^4 – 16x^3 + 12x^2$ and $6x^3 + 6x^2 – 72x$

Solution: (i) $x^2 – 4 = (x + 2) (x – 2)$

$x^2 + 4x + 4 = (x + 2)^2$

HCF of numerical coefficients = 1

HCF of other factors = $(x + 2)^1 = x + 2$

Hence, the required HCF = $x + 2$

(ii) $4x^4 – 16x^3 + 12 x^2 = 4x^2 (x^2 – 4x + 3)$

$= 4x^2 (x – 1) (x – 3)$

$6x^3 + 6x^2 - 72x \qquad = 6x\,(x^2 + x - 12)$

$= 6x\,(x + 4)\,(x - 3)$

Required HCF $= 2x(x - 3)$ [Since HCF of numerical coefficient is 2)

$= 2x^2 - 6x$

(2) LCM of Polynomials

Like HCF, you are also familiar with the LCM (Lowest Common Multiple or Least Common Multiple) of natural numbers in arithmetic. It is the smallest number which is a multiple of each of the given numbers. For instance, the LCM of 8 and 12 is 24 since 24 is the smallest among common multiples of 8 and 12 as given below:

Multiples of 8: 8, 16, <u>24</u>, 32, 40, <u>48</u>, 56, 64, <u>72</u>, 80, ...

Multiples of 12: 12, <u>24</u>, 36, <u>48</u>, 60, <u>72</u>, 84, 96, ...

Common multiple of 8 and 12: 24, 48, 72, ...

On similar lines in Algebra, *the Lowest Common Multiple (LCM) of two or more polynomials is the product of the polynomial(s) of the lowest degree and the smallest numerical coefficient which are multiples of the corresponding elements of each of the given polynomials.*

For example, the LCM of $4(x + 1)^2$ and $6(x + 1)^3$ is $12(x + 1)^3$.

The LCM of monomials is found by multiplying the LCM of numerical coefficients of each of the monomials and all variable factors with highest powers. For example, the LCM of $12x^2y^2z$ and $18x^2yz$ is $36x^2y^2z$ since the LCM of 12 and 18 is 36 and highest powers variable factors x, y and z are x^2, y^2 and z respectively.

Let us, now, consider some examples to illustrate.

Example 4.17: Find the LCM of

(i) $4x^2y$ and x^3y^2 (ii) $(x - 2)^3\,(2x - 3)$ and $(x - 2)^2\,(2x - 3)^3$

Solution: (i) LCM of numerical coefficient 4 and 1 is 4.

Since highest power of x is x^3 and that of y is y^2,

the required LCM is $4x^3y^2$

(ii) Obviously LCM of numerical coefficients 1 and 1 is 1.

In the given polynomials, highest power of the factor $(x - 2)$ is $(x - 2)^3$ and that of $(2x - 3)$ is $(2x - 3)^3$.

LCM of the given polynomials $= 1 \times (x - 2)^3 \times (2x - 3)^3$

$= (x - 2)^3\,(2x - 3)^3$

In view of Example 4.17 (ii), we can say that to determine the LCM of polynomials, which can be easily factorised, we express each of the polynomials as the product of factors. Then, the LCM of the given polynomials is the product of the LCM of the numerical coefficients and all other factors with their highest powers which occur in factorization of any of the polynoials. For further clarification, we take Example 4.18 given below.

Example 4.18: Find the LCM of

(i) $(x-2)(x^2-3x+2)$ and x^2-5x+6

(ii) $8(x^3-27)$ and $12(x^5+27x^2)$

Solution: (i) $(x-2)(x^2-3x+2) = (x-2)(x-2)(x-1)$

$= (x-2)^2(x-1)$

Also $x^2-5x+6 = (x-2)(x-3)$

LCM of numerical coefficients = 1

LCM of other factors = $(x-2)^2(x-1)(x-3)$

Hence, the LCM of given polynomials = $(x-1)(x-2)^2(x-3)$

(ii) $8(x^3-27) = 8(x-3)(x^2+3x+9)$

$12(x^5+27x^2) = 12x^2(x^3+27)$

$= 12x^2(x+3)(x^2-3x+9)$

LCM of numerical coefficient 8 and 12 = 24

LCM of other factors = $x^2(x-3)(x+3)(x^2+3x+9)(x^2-3x+9)$

Hence, required LCM = $24x^2(x-3)(x+3)(x^2+3x+9)(x^2-3x+9)$

CHECK YOUR PROGRESS 4.6

1. Find the HCF of the following polynomials:

(i) $27x^4y^2$ and $3xy^3$ (ii) $48y^7x^9$ and $12y^3x^5$

(iii) $(x+1)^3$ and $(x+1)^2(x-1)$ (iv) x^2+4x+4 and $x+2$

(v) $18(x+2)^3$ and $24(x^3+8)$ (vi) $(x+1)^2(x+5)^3$ and $x^2+10x+25$

(vii) $(2x-5)^2(x+4)^3$ and $(2x-5)^3(x-4)$ (viii) x^2-1 and x^4-1

(ix) x^3-y^3 and x^2-y^2 (x) $6(x^2-3x+2)$ and $18(x^2-4x+3)$

2. Find the LCM of the following polynomials:

(i) $25x^3y^2$ and $15xy$ (ii) $30xy^2$ and $48x^3y^4$

(iii) $(x+1)^3$ and $(x+1)^2(x-1)$ (iv) x^2+4x+4 and $x+2$

(v) $18(x+2)^3$ and $24(x^3+8)$ (vi) $(x+1)^2(x+5)^3$ and $x^2+10x+25$

(vii) $(2x-5)^2(x+4)^2$ and $(2x-5)^3(x-4)$ (viii) x^2-1 and x^4-1

(ix) x^3-y^3 and x^2-y^2 (x) $6(x^2-3x+2)$ and $18(x^2-4x+3)$

4.5 RATIONAL EXPRESSIONS

You are already familiar with integers and rational numbers. Just as a number, which can be expressed in the form $\frac{p}{q}$ where p and q ($\neq 0$) are integers, is called a rational number, an algebraic expression, which can be expressed in the from $\frac{P}{Q}$, where P and Q (non-zero polynomials) are polynomials, is called a **rational expression.** Thus, each of the expressions

$$\frac{x+1}{x-1},\ \frac{x^2-3x+5}{x^2-5},\ \frac{\frac{1}{2}a^2+b^2-\frac{5}{6}}{a+b},\ \frac{x^2+\sqrt{2}y^2}{\sqrt{3}x-y}$$

is a rational expression in one or two variables.

Notes:

(1) The polynomial 'x^2+1' is a rational expresion since it can be written as $\frac{x^2+1}{1}$ and you have learnt that the constant 1 in the denominator is a polynomial of degree zero.

(2) The polynomial 7 is a rational expresion since it can be written as $\frac{7}{1}$ where both 7 and 1 are polynomials of degree zero.

(3) Obvioulsy a rational expression need not be a polynomial. For example rational expression $\frac{1}{x}\left(=x^{-1}\right)$ is not a polynomial. On the contrary every polynomial is also a rational expression.

None of the expressions $\frac{\sqrt{x}+2}{1-x}$, $x^2+2\sqrt{x}+3$, $\frac{a^{\frac{2}{3}}-\frac{1}{b}}{a^2+ab+b^2}$ is a rational expression.

CHECK YOUR PROGRESS 4.7

1. Which of the following algebraic expressions are rational expressions?

(i) $\frac{2x-3}{4x-1}$ (ii) $\frac{8}{x^2+y^2}$

(iii) $\dfrac{2\sqrt{3}x^2+\sqrt{5}}{\sqrt{7}}$ (iv) $\dfrac{2x^2-\sqrt{x}+3}{6x}$

(v) $200+\sqrt{11}$ (vi) $\left(a+\dfrac{1}{b}\right)\div b^{\frac{1}{3}}$

(vii) $y^3 + 3yz\,(y + z) + z^3$ (vii) $5\div(a+3b)$

2. For each of the following, cite two examples:
   (i) A rational expression is one variable
   (ii) A rational expression is two variables
   (iii) A rational expression whose numerator is a binomial and whose denominator is trinomial
   (iv) A rational expression whose numerator is a constant and whose denominator is a quadratic polynomial
   (v) A rational expression in two variables whose numerator is a polynomial of degree 3 and whose denominator is a polynomial of degree 5
   (vi) An algebraic expression which is not a rational expression

4.6 OPERATIONS ON RATIONAL EXPRESSIONS

Four fundamental operations on rational expressions are performed in exactly the same way as in case of rational numbers.

(1) Addition and Subtraction of Rational Expressions

For observing the analogy between addition of rational numbers and that of rational expressions, we take the following example. Note that the analogy will be true for subtraction, multiplication and division of rational expressions also.

Example 4.19: Find the sum:

(i) $\dfrac{5}{6}+\dfrac{3}{8}$ (ii) $\dfrac{2x+1}{x-1}+\dfrac{x+2}{x+1}$

Solution: (i) $\dfrac{5}{6}+\dfrac{3}{8} = \dfrac{5\times4+3\times3}{24}$ ← LCM of 6 and 8.

$$= \frac{20+9}{24}$$

$$= \frac{29}{24}$$

(ii) $\frac{2x+1}{x-1}+\frac{x+2}{x+1} = \frac{(2x+1)(x+1)+(x+2)(x-1)}{(x-1)(x+1)}$ ← LCM of (x – 1) and (x + 1)

$$= \frac{2x^2+3x+1+x^2+x-2}{x^2-1}$$

$$= \frac{3x^2+4x-1}{x^2-1}$$

Example 4.20: Subtract $\frac{x-1}{x+1}$ from $\frac{3x-2}{3x+1}$

Solution: $\frac{3x-2}{3x+1}-\frac{x-1}{x+1} = \frac{(x+1)(3x-2)-(x-1)(3x+1)}{(3x+1)(x+1)}$

$$= \frac{3x^2+x-2-(3x^2-2x-1)}{3x^2+4x+1}$$

$$= \frac{3x-1}{3x^2+4x+1}$$

Note: Observe that the sum and difference of two rational expressions are also rational expressions.

Since the sum and difference of two rational expressions are rational expressions, $x+\frac{1}{x}(x \neq 0)$ and $x-\frac{1}{x}(x \neq 0)$ are both rational expressions as x and $\frac{1}{x}$ are both rational expressions. Similarly, each of $x^2+\frac{1}{x^2}, x^3+\frac{1}{x^3}, x^2-\frac{1}{x^2}, x^3-\frac{1}{x^3}$, etc. is a rational expression. These expresions create interest as for given value of $x+\frac{1}{x}$ or $x-\frac{1}{x}$, we can determine values of $x^2+\frac{1}{x^2}, x^2-\frac{1}{x^2}, x^3+\frac{1}{x^3}, x^3-\frac{1}{x^3}$ etc. and in some case vice versa also. Let us concentrate on the following example.

Example 4.21: Find the value of

(i) $x^2+\frac{1}{x^2}$ if $x-\frac{1}{x}=1$ (ii) $x^4+\frac{1}{x^4}$ if $x+\frac{1}{x}=4$

(iii) $x-\frac{1}{x}$ if $x^4+\frac{1}{x^4}=119$ (iv) $x^3+\frac{1}{x^3}$ if $x+\frac{1}{x}=3$

(v) $x^3 - \frac{1}{x^3}$ if $x - \frac{1}{x} = 5$

Solution: (i) We have $x - \frac{1}{x} = 1$

$$\therefore \left(x - \frac{1}{x}\right)^2 = (1)^2$$

$$\Rightarrow x^2 + \frac{1}{x^2} - 2 \times x \times \frac{1}{x} = 1$$

$$\Rightarrow x^2 + \frac{1}{x^2} - 2 = 1$$

Hence, $x^2 + \frac{1}{x^2} = 3$

(ii) $x + \frac{1}{x} = 4$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = (4)^2$$

$$\Rightarrow x^2 + \frac{1}{x^2} + 2 = 16$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 14$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = (14)^2$$

$$\Rightarrow x^4 + \frac{1}{x^4} + 2 = 196$$

So, $x^4 + \frac{1}{x^4} = 194$

(iii) We have $x^4 + \frac{1}{x^4} = 119$

$$\Rightarrow (x^2)^2 + \left(\frac{1}{x^2}\right)^2 + 2 = 119 + 2 = 121$$

$$\Rightarrow \left(x^2+\frac{1}{x^2}\right)^2=(11)^2$$

$$\Rightarrow x^2+\frac{1}{x^2}=11 \qquad \text{[since both } x^2 \text{ and } \frac{1}{x^2} \text{ are positive]}$$

$$\Rightarrow x^2+\frac{1}{x^2}-2=9$$

$$\Rightarrow \left(x-\frac{1}{x}\right)^2=(3)^2$$

$$\therefore x-\frac{1}{x}=\pm 3$$

(iv) We have $x+\frac{1}{x}=3$

$$\therefore \left(x+\frac{1}{x}\right)^3=(3)^3$$

$$\Rightarrow x^3+\frac{1}{x^3}+3\times x\times\frac{1}{x}\left(x+\frac{1}{x}\right)=27$$

$$\Rightarrow x^3+\frac{1}{x^3}+3(3)=27$$

$$\therefore x^3+\frac{1}{x^3}=18$$

(v) We have $x-\frac{1}{x}=5$

$$\therefore \left(x-\frac{1}{x}\right)^3=(5)^3$$

$$\Rightarrow x^3-\frac{1}{x^3}-3\times x\times\frac{1}{x}\left(x-\frac{1}{x}\right)=125$$

$$\Rightarrow x^3-\frac{1}{x^3}-3(5)=125$$

$$\therefore x^3-\frac{1}{x^3}=140$$

CHECK YOUR PROGRESS 4.8

1. Find the sum of rational expressions:

(i) $\frac{x^2+1}{x-2}$ and $\frac{x^2-1}{x-2}$ (ii) $\frac{x+2}{x+3}$ and $\frac{x-1}{x-2}$

(iii) $\frac{x+1}{(x-1)^2}$ and $\frac{1}{x+1}$ (iv) $\frac{3x+2}{x^2-16}$ and $\frac{x-5}{(x+4)^2}$

(v) $\frac{x-2}{x+3}$ and $\frac{x+2}{x+3}$ (vi) $\frac{x+2}{x-2}$ and $\frac{x-2}{x+2}$

(vii) $\frac{x+1}{x+2}$ and $\frac{x^2-1}{x^2+1}$ (vii) $\frac{3\sqrt{2}x+1}{3x^2}$ and $\frac{-2\sqrt{2}x+1}{2x^2}$

2. Subtract

(i) $\frac{x-1}{x-2}$ from $\frac{x+4}{x+2}$ (ii) $\frac{2x-1}{2x+1}$ from $\frac{2x+1}{2x-1}$

(iii) $\frac{1}{x}$ from x (iv) $\frac{2}{x}$ from $\frac{x+1}{x^2-1}$

(v) $\frac{x^2+1}{x-4}$ from $\frac{2x^2+3}{x-4}$ (vi) $\frac{1}{x^2+2}$ from $\frac{2x^3+x^2+3}{(x^2+2)^2}$

(vii) $\frac{x+2}{2(x^2-9)}$ from $\frac{x-2}{(x+3)^2}$ (vii) $\frac{x+1}{x-1}$ from $\frac{4x}{x^2-1}$

3. Find the value of

(i) $a^2+\frac{1}{a^2}$ when $a+\frac{1}{a}=2$ (ii) $a^2+\frac{1}{a^2}$ when $a-\frac{1}{a}=2$

(iii) $a^3+\frac{1}{a^3}$ when $a+\frac{1}{a}=2$ (iv) $a^3+\frac{1}{a^3}$ when $a+\frac{1}{a}=5$

(v) $a^3-\frac{1}{a^3}$ when $a-\frac{1}{a}=\sqrt{5}$ (vi) $8a^3+\frac{1}{27a^3}$ when $2a+\frac{1}{3a}=5$

(vii) $a^3+\frac{1}{a^3}$ when $a+\frac{1}{a}=\sqrt{3}$ (viii) $a^3+\frac{1}{a^3}$ when $a^2+\frac{1}{a^2}=7, a>0$

(ix) $a - \frac{1}{a}$ when $a^4 + \frac{1}{a^4} = 727$ (x) $a^3 - \frac{1}{a^3}$ when $a^4 + \frac{1}{a^4} = 34, a > 0$

(2) Multiplication and Division of Rational Expressions

You know that the product of two rational numbers, say, $\frac{2}{3}$ and $\frac{5}{7}$ is given as $\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$. Similarly, the product of two rational expressions, say, $\frac{P}{Q}$ and $\frac{R}{S}$ where P, Q, R, S (Q, S $\neq$ 0) are polynomials is given by $\frac{P}{Q} \times \frac{R}{S} = \frac{PR}{QS}$. You may observe that the product of two rational expressions is again a rational expression.

Example 4.22: Find the product:

(i) $\frac{5x+3}{5x-1} \times \frac{2x-1}{x+1}$ (ii) $\frac{2x+1}{x-1} \times \frac{x-1}{x+3}$

(iii) $\frac{x^2-7x+10}{(x-4)^2} \times \frac{x^2-7x+12}{x-5}$

Solution: (i) $\frac{5x+3}{5x-1} \times \frac{2x-1}{x+1} = \frac{(5x+3)(2x-1)}{(5x-1)(x+1)}$

$$= \frac{10x^2+x-3}{5x^2+4x-1}$$

(ii) $\frac{2x+1}{x-1} \times \frac{x-1}{x+3} = \frac{(2x+1)(x-1)}{(x-1)(x+3)}$

$= \frac{2x+1}{x+3}$ [Cancelling common factor (x – 1) from numerator and denominator]

(iii) $\frac{x^2-7x+10}{(x-4)^2} \times \frac{x^2-7x+12}{x-5} = \frac{(x^2-7x+10)(x^2-7x+12)}{(x-4)^2(x-5)}$

$$= \frac{(x-2)(x-5)(x-3)(x-4)}{(x-4)^2(x-5)}$$

$$= \frac{(x-2)(x-3)}{(x-4)}$$

[Cancelling common factor (x –4) (x – 5) from numerator and denominator]

$$= \frac{x^2 - 5x + 6}{x - 4}$$

Note: The result (product) obtained after cancelling the HCF from its numerator and denominator is called the result (product) **in lowest terms** or **in lowest form.**

You are also familiar with the division of a rational number, say, $\frac{2}{3}$ by a rational number, say, $\frac{5}{7}$ is given as $\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5}$ where $\frac{7}{5}$ is the reciprocal of $\frac{5}{7}$. Similarly, division of a rational expression $\frac{P}{Q}$ by a non-zero rational expression $\frac{R}{S}$ is given by $\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \times \frac{S}{R}$ where P, Q, R, S are polynomials and $\frac{S}{R}$ is the **reciprocal expression** of $\frac{R}{S}$.

Example 4.23: Find the reciprocal of each of the following rational expressions:

(i) $\frac{x^2 + 20}{x^3 + 5x + 6}$ (ii) $-\frac{2y}{y^2 - 5}$ (iii) $x^3 + 8$

Solution: (i) Reciprocal of $\frac{x^2 + 20}{x^3 + 5x + 6}$ is $\frac{x^3 + 5x + 6}{x^2 + 20}$

(ii) Reciprocal of $-\frac{2y}{y^2 - 5}$ is $-\frac{y^2 - 5}{2y} = \frac{5 - y^2}{2y}$

(iii) Since $x^3 + 8 = \frac{x^3 + 8}{1}$, the reciprocal of $x^3 + 8$ is $\frac{1}{x^3 + 8}$

Example 4.24: Divide:

(i) $\frac{x^2 + 1}{x - 1}$ by $\frac{x - 1}{x + 2}$

(ii) $\frac{x^2 - 1}{x^2 - 25}$ by $\frac{x^2 - 4x - 5}{x^2 + 4x - 5}$ and express the result in lowest form.

Solution: (i) $\frac{x^2+1}{x-1} \div \frac{x-1}{x+2} = \frac{x^2+1}{x-1} \times \frac{x+2}{x-1}$

$$= \frac{(x^2+1)(x+2)}{(x-1)^2} = \frac{x^3+2x^2+x+2}{x^2-2x+1}$$

(ii) $\frac{x^2-1}{x^2-25} \div \frac{x^2-4x-5}{x^2+4x-5} = \frac{(x^2-1)(x^2+4x-5)}{(x^2-25)(x^2-4x-5)}$

$$= \frac{(x-1)(x+1)(x+5)(x-1)}{(x-5)(x+5)(x+1)(x-5)}$$

$$= \frac{(x-1)(x-1)}{(x-5)(x-5)}$$

[Cancelling HCF (x+1)(x+5)]

$$= \frac{x^2-2x+1}{x^2-10x+25}$$

The result $\frac{x^2-2x+1}{x^2-10x+25}$ is in lowest form.

CHECK YOUR PROGRESS 4.9

1. Find the product and express the result in lowest terms:

(i) $\frac{7x+2}{2x^2+3x+1} \times \frac{x+1}{7x^2-5x-2}$

(ii) $\frac{x^3+1}{x^4+1} \times \frac{x^3-1}{x^4-1}$

(iii) $\frac{3x^2-15x+18}{2x-4} \times \frac{17x+3}{x^2-6x+9}$

(iv) $\frac{5x-3}{5x+2} \times \frac{x+2}{x+6}$

(v) $\frac{x^2+1}{x-1} \times \frac{x+1}{x^2-x+1}$

(vi) $\frac{x^3+1}{x-1} \times \frac{x-1}{2x}$

(vii) $\frac{x-3}{x-4} \times \frac{x^2-5x+4}{x^2-2x-3}$

(viii) $\frac{x^2-7x+12}{x^2-2x-3} \times \frac{x^2-2x-24}{x^2-16}$

2. Find the reciprocal of each of the following rational expressions:

(i) $\frac{x^2+2}{x-1}$

(ii) $-\frac{3a}{1-a}$

(iii) $-\dfrac{7}{1-2x-x^2}$ (iv) x^4+1

3. Divide and express the result as a rational expression in lowest terms:

(i) $\dfrac{x^2+11x+18}{x^2-4x-117} \div \dfrac{x^2+7x+10}{x^2-12x-13}$ (ii) $\dfrac{6x^2+x-1}{2x^2-7x-15} \div \dfrac{4x^2+4x+1}{4x^2-9}$

(iii) $\dfrac{x^2+x+1}{x^2-9} \div \dfrac{x^3-1}{x^2-4x+3}$ (iv) $\dfrac{x^2+2x-24}{x^2-x-12} \div \dfrac{x^2-x-6}{x^2-9}$

(v) $\dfrac{3x^2+14x-5}{x^2-3x+2} \div \dfrac{3x^2+2x-1}{3x^2-3x-2}$ (vi) $\dfrac{2x^2+x-3}{(x-1)^2} \div \dfrac{2x^2+5x+3}{x^2-1}$

LET US SUM UP

- Special products, given below, occur very frequently in algebra:

  (i) $(x+y)^2 = x^2 + 2xy + y^2$ (ii) $(x-y)^2 = x^2 - 2xy + y^2$

  (iii) $(x+y)(x-y) = x^2 - y^2$ (iv) $(x+a)(x+b) = x^2 + (a+b)x + ab$

  (v) $(ax+b)(cx+d) = acx^2 + (ad+bc)x + bd$

  (vi) $(x+y)^3 = x^3 + 3xy(x+y) + y^3$ (vii) $(x-y)^3 = x^3 - 3xy(x-y) - y^3$

  (viii) $(x+y)(x^2-xy+y^2) = x^3 + y^3$ (ix) $(x-y)(x^2+xy+y^2) = x^3 - y^3$

- Factorization of a polynomial is a process of writing the polynomial as a product of two (or more) polynomials. Each polynomial in the product is called a factor of the given polynomial.
- A polynomial is said to be completely factorised if it is expressed as a product of factors, which have no factor other than itself, its negative, 1 or –1.
- Apart from the factorization based on the above mentioned special products, we can factorise a polynomial by taking monomial factor out which is common to some or all of the terms of the polynomial using distributive laws.
- HCF of two or more given polynomials is the product of the polynomial of the highest degree and greatest numerical coefficient each of which is a factor of each of the given polynomials.
- LCM of two or more given polynomials is the product of the polynomial of the lowest degree and the smallest numerical coefficient which are multiples of corresponding elements of each of the given polynomials.

- An algebraic expression, which can be expressed in the form $\frac{P}{Q}$ where P and Q are polynomials, Q being a non-zero polynomial, is called a rational expression.
- Operations on rational expressions are performed in the way, they are performed in case of rational numbers. Sum, Difference, Product and Quotient of two rational expressions are also rational expressions.
- Expressing a rational expression into lowest terms means cancellation of common factor, if any, from the numerator and denominator of the rational exprssion.

TERMINAL EXERCISE

1. Mark a tick ☑ against the correct alternative:

(i) If $120^2 - 20^2 = 25p$, then p is equal to

(A) 16 (B) 140 (C) 560 (D) 14000

(ii) $(2a^2 + 3)^2 - (2a^2 - 3)^2$ is equal to

(A) $24a^2$ (B) $24a^4$ (C) $72a^2$ (D) $72a^4$

(iii) $(a^2 + b^2)^2 + (a^2 - b^2)^2$ is equal to

(A) $2(a^2 + b^2)$ (B) $4(a^2 + b^2)$

(C) $4(a^4 + b^4)$ (D) $2(a^4 + b^4)$

(iv) If $m - \frac{1}{m} = -\sqrt{3}$, then $m^3 - \frac{1}{m^3}$ is equal to

(A) 0 (B) $6\sqrt{3}$ (C) $-6\sqrt{3}$ (D) $-3\sqrt{3}$

(v) $\frac{327 \times 327 - 323 \times 323}{327 + 323}$ is equal to

(A) 650 (B) 327 (C) 323 (D) 4

(vi) $8m^3 - n^3$ is equal to:

(A) $(2m - n)(4m^2 - 2mn + n^2)$ (B) $(2m - n)(4m^2 + 2mn + n^2)$

(C) $(2m - n)(4m^2 - 4mn + n^2)$ (D) $(2m - n)(4m^2 + 4mn + n^2)$

(vii) $\frac{467 \times 467 \times 467 + 533 \times 533 \times 533}{467 \times 467 - 467 \times 533 + 533 \times 533}$ is equal to

(A) 66 (B) 198 (C) 1000 (D) 3000

(viii) The HCF of $36a^5b^2$ and $90a^3b^4$ is

(A) $36a^3b^2$ (B) $18a^3b^2$

(C) $90a^3b^4$ (D) $180a^5b^4$

(ix) The LCM of $x^2 - 1$ and $x^2 - x - 2$ is

(A) $(x^2 - 1)(x - 2)$ (B) $(x^2 - 1)(x + 2)$

(C) $(x - 1)^2 (x + 2)$ (D) $(x + 1)^2 (x - 2)$

(x) Which of the following is not a rational expression?

(A) $\sqrt{33}$ (B) $x + \frac{1}{\sqrt{5x}}$

(C) $8\sqrt{x} + 6\sqrt{y}$ (D) $\frac{x - \sqrt{3}}{x + \sqrt{3}}$

2. Find each of the following products:

(i) $(a^m + a^n)(a^m - a^n)$ (ii) $(x + y + 2)(x - y + 2)$

(iii) $(2x + 3y)(2x + 3y)$ (iv) $(3a - 5b)(3a - 5b)$

(v) $(5x + 2y)(25x^2 - 10xy + 4y^2)$ (vi) $(2x - 5y)(4x^2 + 10xy + 25y^2)$

(vii) $\left(a + \frac{5}{4}\right)\left(a + \frac{4}{5}\right)$ (viii) $(2z^2 + 3)(2z^2 - 5)$

(ix) $99 \times 99 \times 99$ (x) $103 \times 103 \times 103$

(xi) $(a + b - 5)(a + b - 6)$ (xii) $(2x + 7z)(2x + 5z)$

3. If $x = a - b$ and $y = b - c$, show that

$(a - c)(a + c - 2b) = x^2 - y^2$

4. Find the value of $64x^3 - 125z^3$ if $4x - 5z = 16$ and $xz = 12$.

5. Factorise:

(i) $x^7 y^6 + x^{22} y^{20}$ (ii) $3a^5b - 243ab^5$

(iii) $3a^6 + 12a^4b^2 + 12a^2b^4$ (iv) $a^4 - 8a^2b^3 + 16b^6$

(v) $3x^4 + 12y^4$ (vi) $x^8 + 14x^4 + 81$

(vii) $x^2 + 16x + 63$ (viii) $x^2 - 12x + 27$

(ix) $7x^2 + xy - 6y^2$ (x) $5x^2 - 8x - 4$

(xi) $x^6 - 729y^6$ (xii) $125a^6 + 64b^6$

6. Find the HCF of

(i) $x^3 - x^5$ and $x^4 - x^7$

(ii) $30(x^2 - 3x + 2)$ and $50(x^2 - 2x + 1)$

7. Find the LCM of

(i) $x^3 + y^3$ and $x^2 - y^2$

(ii) $x^4 + x^2y^2 + y^4$ and $x^2 + xy + y^2$

8. Perform the indicated operation:

(i) $\dfrac{x+1}{(x-1)^2} + \dfrac{1}{x+1}$

(ii) $\dfrac{2x^2+2x-7}{x^2+x-6} - \dfrac{x-1}{x-2}$

(iii) $\dfrac{x-1}{x-2} \times \dfrac{3x+1}{x^2-4}$

(iv) $\dfrac{x^2-1}{x^2-25} \div \dfrac{x^2-4x-5}{x^2+4x-5}$

9. Simplify: $\dfrac{2}{a-1} - \dfrac{2}{a+1} - \dfrac{4}{a^2+1} - \dfrac{8}{a^4+1}$

[Hint : $\dfrac{2}{a-1} - \dfrac{2}{a+1} = \dfrac{4}{a^2-1}$; now combine next term and so on]

10. If $m = \dfrac{x+1}{x-1}$ and $n = \dfrac{x-1}{x+1}$, find $m^2 + n^2 - mn$.

ANSWERS TO CHECK YOUR PROGRESS

4.1

1. (i) $25x^2 + 20xy + y^2$ (ii) $x^2 - 6x + 9$ (iii) $a^2b^2 + 2abcd + c^2d^2$

(iv) $4x^2 - 20xy + 5y^2$ (v) $\dfrac{x^2}{9} + \dfrac{2}{3}x + 1$ (vi) $\dfrac{z^2}{4} - \dfrac{1}{3}z + \dfrac{1}{9}$

(vii) $a^4 - 25$ (viii) $x^2y^2 - 1$ (ix) $x^2 + \dfrac{25}{12}x + 1$

(x) $\frac{4}{9}x^4 - \frac{25}{9}x^2 - 1$ (xi) $6x^2 + 13xy + 6y^2$ (xii) $21x^2 + 8xy - 5y^2$

2. (i) $40x^2$ (ii) $2a^6 + 18$ (iii) $2(a^2x^2 + b^2y^2)$ (iv) $32p^2q^2$

3. (i) 10404 (ii) 11664 (iii) 4761 (iv) 996004
   (v) 6384 (vi) 22451 (vii) 89964 (viii) 249936
   (ix) 11445 (x) 5621 (xi) 8930 (xii) 989028

4.2

1. (i) $27x^3 + 36x^2y + 36xy^2 + 64y^3$ (ii) $p^3 - 3p^2qr + 3pq^2r^2 - q^3r^3$

   (iii) $a^3 + a^2b + \frac{ab^2}{3} + \frac{b^3}{27}$ (iv) $\frac{a^3}{27} - \frac{a^2b}{3} + ab^2 - b^3$

   (v) $\frac{a^6}{8} + \frac{1}{2}a^4b^2 + \frac{2}{3}a^2b^4 + \frac{8}{27}b^6$ (vi) $\frac{a^6x^9}{27} - \frac{2}{3}a^4b^3x^6y^2 + 4a^2b^6x^3y^4 - 8b^9y^6$

2. (i) 512 (ii) 1728 (iii) 5832 (iv) 12167 (v) 148877
   (vi) 110592 (vii) 357911 (viii) 328509 (ix) 912663 (x) 970299

3. (i) $8x^3 + y^3$ (ii) $x^3 - 8$ (iii) $x^3 + 1$

   (iv) $8y^3 - 27z^6$ (v) $64x^3 + 27y^3$ (vi) $27x^3 - \frac{1}{343}y^3$

4. (i) 100 (ii) 1115

5. (i) 15616 (ii) $\frac{27027}{125}$

6. (i) $120x^2 + 250$ (ii) $1000y^3$ (iii) $19x^3 - 19y^3$ (iv) $-117x^3 - 126$

7. (i) 1000 (ii) 444

4.3

1. $5x(2y - 3z)$ 2. $abc(c - b)$
3. $3p(2p - 5q + 9)$ 4. $(b - c)(a^2 - b)$
5. $(4x - y)^2(8ax - 2ay - b)$ 6. $x(x + y)(x^2 - xy + y^2)$
7. $25(2 + 5p)(2 - 5p)$ 8. $(1 + 16y^4)(1 + 4y^2)(1 + 2y)(1 - 2y)$
9. $(5x + 1)(1 - x)$ 10. $(a^2 + bc + ab + ac)(a^2 + bc - ab - ac)$

11. $(5x + 6y - 1)(5x - 6y - 1)$ 12. $(7x - y + 1)(7x - y - 1)$

13. $(m + 7)^2$ 14. $(2x - 1)^2$

15. $(6a + 5)^2$ 16. $(x^3 - 4)^2$

17. $(a^4 + 7a^2 + 1)(a^2 + 3a + 1)(a^2 - 3a + 1)$

18. $(2a^2 + 6ab + 9b^2)(2a^2 - 6ab + 9b^2)$

19. $(x^2 + 2x + 2)(x^2 - 2x + 2)$

20. $(3a^2 + 5a + 4)(3a^2 - 5a + 4)$ 21. (i) 40 (ii) 57200

4.4

1. $(a + 6b)(a^2 - 6ab + 36b^2)$ 2. $(a - 7)(a^2 + 7a + 49)$

3. $(x + 4y)^3$ 4. $(2x - 3y)^3$

5. $(2x - 5y)^3$ 6. $(4k - 3)^3$

7. $(9x^2 - 2)(81x^4 + 18x^2 + 4)$ 8. $x^2(1 + y^2)(1 - y^2 + y^4)$

9. $2a(2a^2 - 3b^2)(4a^2 + 6a^2b^2 + 9b^4)$ 10. $(3b - a - 1)(9b^2 + 3ab + 3b + a^2 + a + 1)$

11. $(2a - 3b + 4c)(4a^2 + 9b^2 - 6ab - 8ac + 12bc + 16c^2$

12. $(4x - 2y + 1)(16x^2 + 8xy - 4x + 4y^2 - 4y + 1)$

4.5

1. $(x + 3)(x + 8)$ 2. $(x - 6y)(x - 9y)$ 3. $(x + 3)(2x - 1)$

4. $2(x - 2y)(3x + y)$ 5. $(2x^2 + 1)(x + 1)(x - 1)$ 6. $(x + 15y)(x - 2y)$

7. $(x + 2)(2x + 7)$ 8. $(2y - 3)(5y - 2)$ 9. $(x - 1)(2x + 1)$

10. $(12 - m)(m + 9)$ 11. $(2a - b - 6)(2a - b + 5)$ 12. $(9y - 7)(5x + y)$

4.6

1. (i) $3xy^2$ (ii) $12y^3x^5$ (iii) $(x + 1)^2$ (iv) $x + 2$ (v) $6(x + 2)$

(vi) $(x + 5)^2$ (vii) $(2x - 5)^2$ (viii) $x^2 - 1$ (ix) $x - y$ (x) $6(x - 1)$

2. (i) $75x^3y^2$ (ii) $240x^3y^4$ (iii) $(x - 1)(x + 1)^3$

(iv) $x^2 + 4x + 4$ (v) $72(x + 2)^3(x^2 - 2x + 4)$ (vi) $(x + 1)^2(x + 5)^3$

(vii) $(x - 4)(x + 4)^2(2x - 5)^3$ (viii) $x^4 - 1$ (ix) $(x - 1)(x + 1)(x^2 + x + 1)$

(x) $18(x - 1)(x - 2)(x - 3)$

4.7

1. (i), (ii), (iii), (v), (vii) and (viii)

4.8

1. (i) $\frac{2x^2}{x-2}$ (ii) $\frac{2x^2+2x-7}{x^2+x-6}$ (iii) $\frac{2x^2+2}{x^3-x^2-x+1}$

   (iv) $\frac{4x^2+5x+28}{x^3+4x^2-16x+64}$ (v) $\frac{2x}{x+3}$ (vi) $\frac{2x^2+8}{x^2-4}$

   (vii) $\frac{2x^3+3x^2-1}{x^3+2x^2+x+2}$ (viii) $\frac{5}{6x^2}$

2. (i) $\frac{x-6}{x^2-4}$ (ii) $\frac{8x}{4x^2-1}$ (iii) $\frac{x^2-1}{x}$

   (iv) $\frac{2-x}{x^2-x}$ (v) $\frac{x^2+2}{x-4}$ (vi) $\frac{2x^3+1}{(x^2+2)^2}$

   (vii) $\frac{x^2-15x+16}{2(x^3+3x^2-9x-27)}$ (viii) $\frac{1-x}{1+x}$

3. (i) 2 (ii) 6 (iii) 2 (iv) 110 (v) $8\sqrt{15}$

   (vi) 115 (vii) 0 (viii) 18 (ix) ± 5 (x) 14

4.9

1. (i) $\frac{1}{2x^2-x-1}$ (ii) $\frac{x^4+x^2+1}{x^6+x^4+x^2+1}$ (iii) $\frac{51x+9}{2x-6}$

   (iv) $\frac{5x^2+7x-6}{5x^2+32x+12}$ (v) $\frac{x^3+x^2+x+1}{x^3-2x^2+2x-1}$ (vi) $\frac{x^3+1}{2x}$

   (vii) $\frac{x-1}{x+1}$ (viii) $\frac{x-6}{x+1}$

2. (i) $\frac{x-1}{x^2+2}$ (ii) $\frac{a-1}{3a}$ (iii) $\frac{x^2+2x-1}{7}$ (iv) $\frac{1}{x^4+1}$

3. (i) $\frac{x+1}{x+5}$ (ii) $\frac{6x^2-11x+3}{2x^2-9x-5}$ (iii) $\frac{1}{x+3}$

(iv) $\frac{x+6}{x+2}$ (v) $\frac{2x^2+11x+5}{x^2-1}$ (vi) 1

ANSWERS TO TERMINAL EXERCISE

1. (i) C (ii) A (iii) D (iv) A (v) D (vi) B (vii) C (viii) B (ix) A (x) C

2. (i) $a^{2m} - a^{2n}$ (ii) $x^2 - y^2 + 4x + 4$ (iii) $4x^2 + 12xy + 9y^2$

(iv) $9a^2 - 30ab + 25b^2$ (v) $125x^3 + 8y^3$ (vi) $8x^3 - 125y^3$

(vii) $a^2 + \frac{41}{20}a + 1$ (viii) $4z^4 - 4z^2 - 15$ (ix) 970299

(x) 1092727 (xi) $a^2 + 2ab - 11a + 30$ (xii) $4x^2 + 24xz + 35z^2$

4. 15616

5. (i) $x^7y^6(1 + x^{15}y^{14})$ (ii) $3ab(a - 3b)(a + 3b)(a^2 + 9b^2)$

(iii) $3a^2(a^2 + 2b^2)^2$ (iv) $(a^2 - 4b^3)^2$

(v) $3(x^2 + 2xy + 2y^2)$ (vi) $(x^4 - 2x^2 + 9)(x^4 + 2x^2 + 9)$

(vii) $(x + 9)(x + 7)$ (viii) $(x - 3)(x - 9)$

(ix) $(x + y)(7x - 6y)$ (x) $(x - 2)(5x + 2)$

(xi) $(x - 3y)(x + 3y)(x^2 - 3xy + 9y^2)(x^2 + 3xy + 9y^2)$

(xii) $(5a^2 + 4b^2)(25a^4 - 20a^2b^2 + 16b^4)$

6. (i) $x^3(1 - x)$ (ii) $10(x - 1)$

7. (i) $(x^2 - y^2)(x^2 - xy + y^2$ (ii) $x^4 + x^2y^2 + y^4$

8. (i) $\frac{2x^2+2}{x^3-x^2-x+1}$ (ii) $\frac{x+2}{x+3}$

(iii) $\frac{3x^2-2x-1}{x^3+2x^2-4x-8}$ (iv) $\frac{x^2-2x+1}{x^2-10x+25}$

9. $\frac{16}{a^8-1}$

10. $\frac{x^4+14x^2+1}{x^4-2x^2+1}$