21

DETERMINANTS

Every square matrix is associated with a unique number called the determinant of the matrix.

In this lesson, we will learn various properties of determinants and also evaluate determinants by different methods.

OBJECTIVES

After studying this lesson, you will be able to :

- define determinant of a square matrix;
- define the minor and the cofactor of an element of a matrix;
- find the minor and the cofactor of an element of a matrix;
- find the value of a given determinant of order not exceeding 3;
- state the properties of determinants;
- evaluate a given determinant of order not exceeding 3 by using expansion method;

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of solution of equations
- Knowledge of number system including complex number
- Four fundamental operations on numbers and expressions

21.1 DETERMINANT OF ORDER 2

Let us consider the following system of linear equations:

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

On solving this system of equations for x and y, we get

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \text{ and } y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \text{ provided } a_1b_2 - a_2b_1 \neq 0$$

The number $a_1b_2 - a_2b_1$ determines whether the values of x and y exist or not.

The number $a_1b_2 - a_2b_1$ is called the value of the determinant, and we write

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

i.e. a_{11} belongs to the 1st row and 1st column
a_{12} belongs to the 1st row and 2nd column
a_{21} belongs to the 2nd row and 1st column
a_{22} belongs to the 2nd row and 2nd column

21.2 EXPANSION OF A DETERMINANT OF ORDER 2

A formal rule for the expansion of a determinant of order 2 may be stated as follows:

In the determinant, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

write the elements in the following manner :

$$\begin{matrix} a_{1} & & a_{12} \\ & \times & \\ a_{21} & & a_{22} \end{matrix}$$

Multiply the elements by the arrow. The sign of the arrow going **downwards** is positive, i.e., a_{11} a_{22} and the sign of the arrow going **upwards** is negative, i.e., $-a_{21}$ a_{12}
Add these two products, i.e., $a_{11}\,a_{22} + (-a_{21}.a_{12})$ or $a_{11}\,a_{22} - a_{21}\,a_{12}$ which is the required value of the determinant.

Example 21.1 Evaluate :

(i) $\begin{vmatrix} 6 & 4 \\ 8 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} a+b & 2b \\ 2a & a+b \end{vmatrix}$ (iii) $\begin{vmatrix} x^2+x+1 & x+1 \\ x^2-x+1 & x-1 \end{vmatrix}$

Solution :

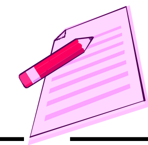
(i) $\begin{vmatrix} 6 & 4 \\ 8 & 2 \end{vmatrix} = (6 \times 2) - (8 \times 4) = 12 - 32 = -20$

(ii) $\begin{vmatrix} a+b & 2b \\ 2a & a+b \end{vmatrix} = (a+b)\,(a+b) - (2a)\,(2b)$

$= a^2+2ab+b^2 - 4ab = a^2+b^2 - 2ab = (a-b)^2$

(iii) $\begin{vmatrix} x^2+x+1 & x+1 \\ x^2-x+1 & x-1 \end{vmatrix} = (x^2+x+1)\,(x-1) - (x^2-x+1)\,(x+1)$

$= (x^3-1) - (x^3+1) = -2$



Notes

Example 21.2 Find the value of x if

(i) $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = 6$ (ii) $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = 0$

Solution :

(i) Now, $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = (x-3)(x+3) - x(x+1)$

$= (x^2 - 9) - x^2 - x = -x - 9$

According to the question,

$-x - 9 = 6$

$\Rightarrow \quad x = -15$

(ii) Now, $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = (2x-1)(4x+2) - (x+1)(2x+1)$

$= 8x^2 + 4x - 4x - 2 - 2x^2 - x - 2x - 1$

$= 6x^2 - 3x - 3 = 3(2x^2 - x - 1)$

According to the equation

$3(2x^2 - x - 1) = 0$

or, $2x^2 - x - 1 = 0$

or, $2x^2 - 2x + x - 1 = 0$

or, $2x(x - 1) + 1(x - 1) = 0$

or, $(2x+1)(x - 1) = 0$

or, $x = 1, -\frac{1}{2}$

21.3 DETERMINANT OF ORDER 3

The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ contains nine quantities $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3$ and c_3 aranged in 3 rows and 3 columns, is called determinant of order 3 (or a determinant of third order). A determinant of order 3 has $(3)^2 = 9$ elements.

Using double subscript notations, viz., $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ for the elements

$a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3$ and c_3, we write a determinant of order3 as follows: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Usually a determinant, whether of order 2 or 3, is denoted by Δ or |A|, |B| etc.

$$\Delta = |a_{ij}| \text{ , where } i = 1, 2, 3 \text{ and } j = 1, 2, 3$$

21.4 DETERMINANT OF A SQUARE MATRIX

With each square matrix of numbers (we associate) a "determinant of the matrix".

With the 1 $\times$ 1 matrix [a], we associate the determinant of order 1 and with the only element a. The value of the determinant is a.

If A= $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a square matrix of order 2, then the expression $a_{11}a_{22} - a_{21}a_{12}$ is defined as the determinant of order 2. It is denoted by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

With the 3 $\times$ 3 matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, we associate the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and its value is defined to be

$$a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)\, a_{12} \times \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 21.3 If A = $\begin{pmatrix} 3 & 6 \\ 1 & 5 \end{pmatrix}$, find |A|

Solution : $|A| = \begin{vmatrix} 3 & 6 \\ 1 & 5 \end{vmatrix} = 3 \times 5 - 1 \times 6 = 15 - 6 = 9$

Example 21.4 If A= $\begin{pmatrix} a+b & a \\ b & a-b \end{pmatrix}$, find |A|

Solution : $|A| = \begin{vmatrix} a+b & a \\ b & a-b \end{vmatrix}$ = (a+b) (a−b) − b × a = $a^2 - b^2 - ab$

Note : 1. The determinant of a unit matrix I is 1.

2. A square matrix whose determinant is zero, is called the singular matrix.

21.5 EXPANSION OF A DETERMINANT OF ORDER 3

In Section 4.4, we have written

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)a_{12} \times \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \times \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

which can be further expanded as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

We notice that in the above method of expansion, each element of first row is multiplied by the second order determinant obtained by deleting the row and column in which the element lies.

Further, mark that the elements a_{11}, a_{12} and a_{13} have been assigned positive, negative and positive signs, respectively. In other words, they are assigned positive and negative signs, alternatively, starting with positive sign. If the sum of the subscripts of the elements is an even number, we assign positive sign and if it is an odd number, then we assign negative sign. Therefore, a_{11} has been assigned positive sign.

Note : We can expand the determinant using any row or column. The value of the determinant will be the same whether we expand it using first row or first column or any row or column, taking into consideration rule of sign as explained above.

Example 21.5 Expand the determinant, using the first row

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 5 \end{vmatrix}$$

Solution : $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 5 \end{vmatrix} = 1 \times \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} - 2 \times \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} + 3 \times \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$

$= 1 \text{ x } (20 - 2) - 2 \times (10 - 3) + 3 \times (4 - 12)$

$= 18 - 14 - 24$

$= -20$

Example 21.6 Expand the determinant, by using the second column

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

Solution : $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-1) \times 2 \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + (-1)\ 3 \times \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix}$

$= -2 \times (3 - 4) + 1 \times (1 - 6) - 3 \times (2 - 9)$

$= 2 - 5 + 21$

$= 18$

CHECK YOUR PROGRESS 21.1

1. Find |A|, if

(a) $A = \begin{bmatrix} 2+\sqrt{3} & 5 \\ 2 & 2-\sqrt{3} \end{bmatrix}$ (b) $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

(c) $A = \begin{bmatrix} \sin\alpha + \cos\beta & \cos\beta + \cos\alpha \\ \cos\beta - \cos\alpha & \sin\alpha - \sin\beta \end{bmatrix}$ (d) $\begin{bmatrix} a+bi & c+di \\ c-di & a-bi \end{bmatrix}$

2. Find which of the following matrices are singular matrices :

(a) $\begin{bmatrix} 5 & 5 & 1 \\ -5 & 1 & 1 \\ 10 & 7 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 4 & 1 & 5 \end{bmatrix}$

3. Expand the determinant by using first row

(a) $\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 1 & -5 \\ 0 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix}$ (c) $\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}$ (d) $\begin{vmatrix} x & y & z \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix}$

21.6 MINORS AND COFACTORS

21.6.1 *Minor of a_{ij} in |A|*

To each element of a determinant, a number called its minor is associated.

The minor of an element is the value of the determinant obtained by deleting the row and column containing the element.

Thus, the minor of an element a_{ij} in |A| is the value of the determinant obtained by deleting the i^{th} row and j^{th} column of |A| and is denoted by M_{ij}. For example, minor of 3 in the determinant

$\begin{vmatrix} 3 & 2 \\ 5 & 7 \end{vmatrix}$ is 7.

Example 21.7 Find the minors of the elements of the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solution :

Let M_{ij} denote the minor of a_{ij}. Now, a_{11} occurs in the 1^{st} row and 1^{st} column. Thus to find the minor of a_{11}, we delete the 1^{st} row and 1^{st} column of |A|.

The minor M_{11} of a_{11} is given by

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

Similarly, the minor M_{12} of a_{12} is given by

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31} \; ; \quad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{32}a_{13} \; ; \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}a_{33} - a_{31}a_{13}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}$$

Similarly we can find M_{31}, M_{32} and M_{33}.

21.6.2 *Cofactors of a_{ij} in $|A|$*

The cofactor of an element a_{ij} in a determinant is the minor of a_{ij} multiplied by $(-1)^{i+j}$. It is usually denoted by C_{ij}, Thus,

Cofactor of $a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}$

Example 21.8 Find the cofactors of the elements a_{11}, a_{12}, and a_{21} of the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solution :

The cofactor of any element a_{ij} is $(-1)^{i+j} M_{ij}$, then

$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (a_{22}\, a_{33} - a_{32}\, a_{23})$

$= (a_{22}\, a_{33} - a_{32}\, a_{23})$

$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -(a_{21}\, a_{33} - a_{31}\, a_{23}) = (a_{31}a_{23} - a_{21}a_{33})$

and $C_{21} = (-1)^{2+1} M_{21} = -M_{21} = (a_{32}\, a_{13} - a_{12}\, a_{33})$

Example 21.9 Find the minors and cofactors of the elements of the second row in the determinant

$$|A| = \begin{vmatrix} 1 & 6 & 3 \\ 5 & 2 & 4 \\ 7 & 0 & 8 \end{vmatrix}$$

Solution : The elements of the second row are $a_{21}=5$; $a_{22}=2$; $a_{23}=4$.

Minor of a_{21} (i.e., 5) $= \begin{vmatrix} 6 & 3 \\ 0 & 8 \end{vmatrix} = 48 - 0 = 48$

Minor of a_{22} (i.e., 2) $= \begin{vmatrix} 1 & 3 \\ 7 & 8 \end{vmatrix} = 8 - 21 = -13$

and Minor of a_{23} (i.e., 4) $= \begin{vmatrix} 1 & 6 \\ 7 & 0 \end{vmatrix} = 0 - 42 = -42$

The corresponding cofactors will be

$$C_{21} = (-1)^{2+1}M_{21} = -(48) = -48$$

$$C_{22} = (-1)^{2+2}M_{22} = +(-13) = -13$$

and $$C_{23} = (-1)^{2+3}M_{23} = -(-42) = 42$$

CHECK YOUR PROGRESS 21.2

1. Find the minors and cofactors of the elements of the second row of the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$$

2. Find the minors and cofactors of the elements of the third column of the determinat

$$\begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

3. Evaluate each of the following determinants using cofactors:

(a) $\begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 3 & -4 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} 3 & 4 & 5 \\ -6 & 2 & -3 \\ 8 & 1 & 7 \end{vmatrix}$

(d) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ (e) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ (f) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

4. Solve for x, the following equations:

(a) $\begin{vmatrix} x & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 0 & 2 \end{vmatrix} = 0$ (b) $\begin{vmatrix} x & 3 & 3 \\ 3 & 3 & x \\ 2 & 3 & 3 \end{vmatrix} = 0$ (c) $\begin{vmatrix} x^2 & x & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 28$

21.7 PROPERTIES OF DETERMINANTS

We shall now discuss some of the properties of determinants. These properties will help us in expanding the determinants.

Property 1: The value of a determinant remains unchanged if its rows and columns are interchanged.

$$\text{Let } \Delta = \begin{vmatrix} 2 & -1 & 3 \\ 0 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix}$$

Expanding the determinant by first column, we have

$$\Delta \quad = 2\begin{vmatrix} -3 & 0 \\ 2 & -1 \end{vmatrix} - 0\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} + 4\begin{vmatrix} -1 & 3 \\ -3 & 0 \end{vmatrix}$$

$$= 2\,(3-0) - 0\,(1-6) + 4\,(0+9)$$

$$= 6 + 36 = 42$$

Let Δ' be the determinant obtained by interchanging rows and columns of Δ. Then

$$\Delta' = \begin{vmatrix} 2 & 0 & 4 \\ -1 & -3 & 2 \\ 3 & 0 & -1 \end{vmatrix}$$

Expanding the determinant Δ' by second column, we have (Recall that a determinant can be expanded by any of its rows or columns)

$$(-)\,0\begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} + (-)\,0\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$$

$$= 0 + (-3)\,(-2-12) + 0$$

$$= 42$$

Thus, we see that $\Delta = \Delta'$

Property 2: If two rows (or columns) of a determinant are interchanged, then the value of the determinant changes in sign only.

$$\text{Let } \Delta = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

Expanding the determinant by first row, we have

$$= 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 2\,(4-3) - 3\,(2-9) + 1\,(1-6)$$

$$= 2 + 21 - 5 = 18$$

Let Δ' be the determinant obtained by interchanging C_1 and C_2

$$\text{Then } \Delta' = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

Expanding the determinant Δ' by first row, we have

$$3\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 3\,(2-9) - 2(4-3) + 1(6-1)$$

$$= -21 - 2 + 5 = -18$$

Thus we see that $\Delta' = -\Delta$

Corollary

If any row (or a column) of a determinant is passed over 'n' rows (or columns), then the resulting determinant Δ' is $\Delta = (-1)^n\ \Delta$

For example ,
$$\begin{vmatrix} 2 & 3 & 5 \\ 1 & 5 & 6 \\ 0 & 4 & 2 \end{vmatrix} = (-1)^2 \begin{vmatrix} 1 & 5 & 6 \\ 0 & 4 & 2 \\ 2 & 3 & 5 \end{vmatrix}$$

$$= 2\,(10-24) - 3\,(2-0) + 5(4)$$

$$= -28 - 6 + 20 = -14$$

Property 3: If any two rows (or columns) of a determinant are identical then the value of the determinant is zero.

Proof : Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

be a determinant with identical columns C_1 and C_2 and let Δ' determinant obtained from Δ by interchanging C_1 and C_2

Then,

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

which is the same as Δ, but by property 2, the value of the determinant changes in sign, if its any two adjacent rows (or columns) are interchnaged

Therefore $\Delta' = -\Delta$

Thus, we find that

or $2\Delta = 0 \Rightarrow \Delta = 0$

Hence the value of a determinant is zero, if it has two identical rows (or columns).

Property 4: If each element of a row (or column) of a determinant is multiplied by the same constant say, $k \neq 0$, then the value of the determinat is multiplied by that constant k.

$$\text{Let } \Delta = \begin{vmatrix} 2 & 1 & -5 \\ 0 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix}$$

Expanding the determinant by first row, we have

$$\Delta = 2(3-0) - 1\,(0-0) + (-5)\,(0+12\,)$$

$$= 6 - 60 = -54$$

Let us multiply column 3 of Δ by 4. Then, the new determinant Δ' is :

$$\Delta' = \begin{vmatrix} 2 & 1 & -20 \\ 0 & -3 & 0 \\ 4 & 2 & -4 \end{vmatrix}$$

Expanding the determinant Δ' by first row, we have

$$\Delta' = 2\,(12-0) - 1\,(0-0) + (-20)\,(0+12)$$

$$= 24 - 240 = -216$$

$$= 4\,\Delta$$

Corollary :
If any two rows (or columns) of a determinant are proportional, then its value is zero.

$$\text{Proof: Let } \Delta = \begin{vmatrix} a_1 & b_1 & ka_1 \\ a_2 & b_2 & ka_2 \\ a_3 & b_3 & ka_3 \end{vmatrix}$$

Note that elements of column 3 are *k* times the corresponding elements of column 1

$$\text{By Property 4, } \Delta = k\begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix}$$

$$= k \times 0 \qquad \text{(by Property 2)}$$

$$= 0$$

Property 5: If each element of a row (or of a column) of a determinant is expressed as the sum (or difference) of two or more terms, then the determinant can be expressed as the sum (or difference) of two or more determinants of the same order whose remaining rows (or columns) do not change.

$$\text{Proof : Let } \Delta = \begin{vmatrix} a_1 + \alpha & b_1 + \beta & c_1 + \gamma \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then, on expanding the determinant by the first row, we have

$$\Delta = (a_1 + \alpha)(b_2c_3 - b_3c_2) - (b_1 + \beta)(a_2c_3 - a_3c_2) + (c_1 + \gamma)(a_2b_3 - a_3b_2)$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) + \alpha(b_2c_3 - b_3c_2)$$

$$-\beta(a_2c_3 - a_3c_2) + \gamma(a_2b_3 - a_3b_2)$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & \beta & \gamma \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Thus, the determinant Δ can be expressed as the sum of the determinants of the same order.

Property 6: The value of a determinant does not change, if to each element of a row (or a column) be added (or subtracted) the some multiples of the corresponding elements of one or more other rows (or columns)

Proof : Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Δ' be the determinant obtained from Δ by corresponding elements of R_3

i.e. $R_1 \to R_1 + kR_3$

Then, $\Delta' = \begin{vmatrix} a_1 + ka_3 & b_1 + kb_3 & c_1 + kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_3 & kb_3 & kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

or, $\Delta' = \Delta + k \times 0$ (Row 1 and Row 3 are identical)

$\Delta' = \Delta$

21.8 EVALUATION OF A DETERMINANT USING PROPERTIES

Now we are in a position to evaluate a determinant easily by applying the aforesaid properties. The purpose of simplification of a determinant is to make maximum possible zeroes in a row (or column) by using the above properties and then to expand the determinant by that row (or column). We denote 1st, 2nd and 3rd row by R_1, R_2, and R_3 respectively and 1st,2nd and 3rd column by C_1, C_2 and C_3 respectively.

Example 21.10 Show that $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} = 0$

where w is a non-real cube root of unity.

Solution : $\Delta = \begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

Add the sum of the 2nd and 3rd column to the 1st column. We write this operation as $C_1 \rightarrow C_1 + (C_2 + C_3)$

$$\therefore \quad \Delta = \begin{vmatrix} 1+w+w^2 & w & w^2 \\ w+w^2+1 & w^2 & 1 \\ w^2+1+w & 1 & w \end{vmatrix} = \begin{vmatrix} 0 & w & w^2 \\ 0 & w^2 & 1 \\ 0 & 1 & w \end{vmatrix} = 0 \quad \text{(on expanding by } C_1\text{)}$$

(since w is a non-real cube root of unity, therefore, $1+w+w^2 = 0$)

Example 21.11 Show that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution : $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

$$= \begin{vmatrix} 0 & a-c & bc-ab \\ 0 & b-c & ca-ab \\ 1 & c & ab \end{vmatrix} [R_1 \rightarrow R_1 - R_3 \text{ and } R_2 \rightarrow R_2 - R_3]$$

$$= \begin{vmatrix} 0 & a-c & b(c-a) \\ 0 & b-c & a(c-b) \\ 1 & c & ab \end{vmatrix} = (a-c)(b-c)\begin{vmatrix} 0 & 1 & -b \\ 0 & 1 & -a \\ 1 & c & ab \end{vmatrix}$$

Expanding by C_1, we have

$$\Delta = (a-c)(b-c)\begin{vmatrix} 1 & -b \\ 1 & -a \end{vmatrix} = (a-c)(b-c)(b-a)$$

$$= (a-b)(b-c)(c-a)$$

Example 21.12

Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4\,abc$

Solution : $\Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

$$= \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \quad R_1 \to R_1 - (R_2 + R_3)$$

Expanding by R_1, we get

$$= 0\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - (-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} - 2b\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$$= 2c\,[b(a+b) - bc] - 2b[bc - c(c+a)]$$

$$= 2bc\,[a+b-c] - 2bc[b-c-a]$$

$$= 2bc\,[(a+b-c) - (b-c-a)]$$

$$= 2bc\,[a+b-c-b+c+a]$$

$$= 4abc$$

Example 21.13 Evaluate:

$$\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Solution : $\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} \quad C_1 \to C_1 + C_2 + C_3 = 0,$$

Example 21.14 Prove that

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Solution : Δ

$$= \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & bc & bc+ab+ac \\ 1 & ca & ca+bc+ba \\ 1 & ab & ab+ca+cb \end{vmatrix} \quad C_3 \to C_2 + C_3$$

$$= (ab+bc+ca)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$$

$$= (ab+bc+ca) \times 0 \qquad \text{(by Property 3)}$$

$$= 0$$

Example 21.15 Show that

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution : Δ

$$= \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

$$= \text{abc}\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

$$= \text{abc}\begin{vmatrix} -a & b & c \\ 0 & 0 & 2c \\ 0 & 2b & 0 \end{vmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$= abc(-a)\begin{vmatrix} 0 & 2c \\ 2b & 0 \end{vmatrix} \quad \text{(on expanding by } C_1\text{)}$$

$$= abc(-a)(-4bc)$$

$$= 4a^2b^2c^2$$

Example 21.16 Show that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = a^2\,(a+3)$$

Solution : $\Delta \quad = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$

$$= \begin{vmatrix} a+3 & 1 & 1 \\ a+3 & 1+a & 1 \\ a+3 & 1 & 1+a \end{vmatrix} \quad C_1 \to C_1 + C_2 + C_3$$

$$= (a+3)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$$= (a+3)\begin{vmatrix} 1 & 0 & 0 \\ 1 & a & 0 \\ 1 & 0 & a \end{vmatrix} \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$= (a+3) \times (1)\begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix}$$

$$= (a+3)(a^2)$$

$$= a^2(a+3)$$

CHECK YOUR PROGRESS 21.3

1. Show that $\begin{vmatrix} x+3 & x & x \\ x & x+3 & x \\ x & x & x+3 \end{vmatrix} = 27\,(x+1)$

2. Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

3. Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = bc + ca + ab + abc$

4. Show that $\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9b^2(a+b)$

5. Show that $\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2$

6. Show that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

7. Evaluate

(a) $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$ (b) $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

8. Solve for x :

$$\begin{vmatrix} 3x-8 & 3 & x \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

21.11 Application of Determinants

Determinant is used to find area of a triangle.

21.11.1 Area of a Triangle

We know that area of a triangle ABC, (say) whose vertices are $(x_1 y_1)$, $(x_2 y_2)$ and $(x_3 y_3)$ is given by

$$\text{Area of } (\Delta ABC) = \frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right] \qquad ...(i)$$

Also, $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1\begin{vmatrix} y_2 & 1 \\ y_3 & 1 \end{vmatrix} - x_2\begin{vmatrix} y_1 & 1 \\ y_3 & 1 \end{vmatrix} + x_3\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}$ [expanding along C_1]

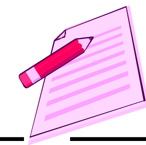
$$= x_1(y_2-y_3)-x_2(y_1-y_3)+x_3(y_1-y_2)$$

$$= x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2) \qquad ...(ii)$$

from (*i*) and (*ii*)

$$\text{Area } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & x_2 & 1 \\ y_3 & y_3 & 1 \end{vmatrix}$$

Thus the area of a triangle having vertices as $(x_1\ y_1)$, $(x_2\ y_2)$ and $(x_3\ y_3)$ is given by



$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & x_2 & 1 \\ y_3 & y_3 & 1 \end{vmatrix}$$

21.11.2 Condition of collinearity of three points :

Let A(x_1 y_1), B(x_2 y_2) and C(x_3 y_3) be three points then

A, B, C are collinear if area of ΔABC = 0

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

21.11.2 Equation of a line passing through the given two points

Let the two points be P(x_1, y_1) and Q(x_2 y_2) and R(x y) be any point on the line joining P and Q since the points P, Q and R are collinear.

$$\therefore \quad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Thus the equation of the line joining points (x_1 y_1) and (x_1 y_2) is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Example 21.17 Find the area of the triangle with vertices P(5, 4), Q(–2, 4) and R(2, –6)

Solution : Let A be the area of the triangle PQR, then

$$A = \frac{1}{2}\begin{vmatrix} 5 & 4 & 1 \\ -2 & 4 & 1 \\ 2 & -6 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[5(4-(-6)) - 4(-2-2) + 1(12-8)\right]$$

$$= \frac{1}{2}[50 + 16 + 4] = \frac{1}{2}(70) = 35 \text{ sq units.}$$

Example 21.18 Show that points (a, b + c), (b, c + a) and (c, a + b) are collinear.

Solution : We have

$$\Delta = \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

$$c_2 \to c_2 + c_1$$

$$= \begin{vmatrix} a & a+b+c & 1 \\ b & b+c+a & 1 \\ c & c+a+b & 1 \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} = (a + b + c) \times 0 = 0$$

Hence, the given points are collinear.

Example 21.19 Find equation of the line joining A(1, 3) and B(2, 1) using determinants.

Solution : Let P(x, y) be any point on the line joining A(1, 3) and B(2, 1). Then

$$\begin{vmatrix} x & y & 1 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad x(3-1) - y(1-2) + 1(1-6) = 0$$

$$\Rightarrow \quad 2x + y - 5 = 0$$

This is the required equation of line AB.

CHECK YOUR PROGRESS 21.4

1. Find area of the ΔABC when A, B and C are (3, 8), (–4, 2) and (5, –1) respectively.
2. Show that points A(5, 5), B(–5, 1) and C(10, 7) are collinear.
3. Using determinants find the equation of the line joining (1, 2) and (3, 6).

LET US SUM UP

- The expression $a_1b_2 - a_2b_1$ is denoted by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$
- With each square matrix, a determinant of the matrix can be associated.
- The *minor* of any element in a determinant is obtained from the given determinant by deleting the row and column in which the element lies.

- The *cofactor* of an element a_{ij} in a determinant is the minor of a_{ij} multiplied by $(-1)^{i+j}$
- A determinant can be expanded using any row or column. The value of the determinant will be the same.
- A square matrix whose determinant has the value zero, is called a *singular matrix*.
- The value of a determinant remains unchanged, if its rows and columns are interchanged.
- If two rows (or columns) of a determinant are interchanged, then the value of the determinant changes in sign only.
- If any two rows (or columns) of a determinant are identical, then the value of the determinant is zero.
- If each element of a row (or column) of a determinant is multiplied by the same constant, then the value of the determinant is multiplied by the constant.
- If any two rows (or columns) of a determinant are proportional, then its value is zero.
- If each element of a row or column from of a determinant is expressed as the sum (or differenence) of two or more terms, then the determinant can be expressed as the sum (or difference) of two or more determinants of the same order.
- The value of a determinant does not change if to each element of a row (or column) be added to (or subtracted from) some multiples of the corresponding elements of one or more rows (or columns).
- Product of a matrix and its inverse is equal to identity matrix of same order.
- Inverse of a matrix is always unique.
- All matrices are not necessarily invertible.
- Three points are collinear if the area of the triangle formed by these three points is zero.

SUPPORTIVE WEB SITES

http://www.math.odu.edu/~bogacki/cgi-bin/lat.cgi?c=det

http://mathworld.wolfram.com/Determinant.html

http://en.wikipedia.org/wiki/Determinant

http://www-history.mcs.st-andrews.ac.uk/HistTopics/Matrices_and_determinants.html

TERMINAL EXERCISE

1. Find all the minors and cofactors of $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix}$

2. Evaluate $\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$ by expanding it using the first column.

3. Evaluate $\begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix}$ 4. Solve for x, if $\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & x \\ 1 & 3 & x \end{vmatrix} = 0$

5. Using properties of determinants, show that

(a) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$

(b) $\begin{vmatrix} 1 & x+y & x^2+y^2 \\ 1 & y+z & y^2+z^2 \\ 1 & z+x & z^2+x^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

6. Evaluate: (a) $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & w^3 & w^5 \\ w^3 & 1 & w^4 \\ w^5 & w^5 & 1 \end{vmatrix}$

, w being an imaginary cube-root of unity

7. Find the area of the triangle with vertices at the points :

(i) (2, 7), (1, 1) and (10, 8) (ii) $(-1,-8), (-2,-3)$ and $(3,2)$

(ii) (0, 0) (6, 0) and (4, 3) (iv) (1, 4), (2, 3) and (–5, –3)

8. Using determinants find the value of k so that the following points become collinear

(i) $(k, 2-2k), (-k+1, 2k)$ and $(-4-k, 6-2k)$

(ii) $(k,-2), (5,2)$ and $(6,8)$

(iii) $(3,-2), (k,2)$ and $(8,8)$

(iv) $(1,-5)(-4,5)(k,7)$

9. Using determinants, find the equation of the line joining the points

(i) (1, 2) and (3, 6) (ii) (3, 1) and (9, 3)

10. If the points (a, 0), (0, b) and (1, 1) are collinear then using determinants show that $ab = a + b$

ANSWERS

CHECK YOUR PROGRESS 21.1

1. (a) 11 (b) 1 (c) 0 (d) $(a^2+b^2)-(c^2+d^2)$

2. (a) and (d)

3. (a) 18 (b) −54

(c) $adf + 2bce - ae^2 - fb^2 - de^2$ (d) $x-1$

CHECK YOUR PROGRESS 21.2

1. $M_{21} = 39;\ C_{21} = -39$

$M_{22} = 3;\ C_{22} = 3$

$M_{23} = -11;\ C_{23} = 11$

2. $M_{13} = -5;\ C_{13} = -5$

$M_{23} = -7;\ C_{23} = 7$

$M_{33} = 1;\ C_{33} = 1$

3. (a) 19 (b) 0 (c) –131

(d) $(a-b)(b-c)(c-a)$ (e) $4abc$ (f) 0

4. (a) $x = 2$ (b) $x = 2, 3$ (c) $x = 2, -\frac{17}{7}$

CHECK YOUR PROGRESS 21.3

7. (a) a^3 (b) $2abc(a+b+c)^3$

8. $x = \frac{2}{3}, \frac{11}{3}, \frac{11}{3}$

CHECK YOUR PROGRESS 21.4

1. $\frac{75}{2}$ sq units (3) $y = 2x$

TERMINAL EXERCISE

1. $M_{11} = -2,\ M_{12} = -1,\ M_{13} = 1,\ M_{21} = -7,\ M_{22} = -5,\ M_{23} = -1,$
$M_{31} = -8,\ M_{32} = -7,\ M_{33} = -2$

$C_{11} = -2,\ C_{12} = 1,\ C_{13} = 1,\ C_{21} = 7,\ C_{22} = -5,\ C_{23} = 1,$
$C_{31} = -8,\ C_{32} = 7,\ C_{33} = -2$

2. 0

3. –31

4. $x = 0,\ x = 1$

6. ( ) -8 (b) 0

7. (*i*) $\frac{45}{2}$ sq units (*ii*) 5 sq units

(*iii*) 9 sq units (*iv*) $\frac{15}{2}$ sq units

8. (*i*) $k = -1, \frac{1}{2}$ (*ii*) $k = \frac{13}{3}$

(*iii*) k = 5 (*iv*) $k = -5$

9. (*i*) $y = 2x$ (ii) $x = 3y$