



## RELATIONS BETWEEN SIDES AND ANGLES OF A TRIANGLE

In earlier lesson, we have learnt about trigonometric functions of real numbers, relations between them, drawn the graphs of trigonometric functions, studied the characteristics from their graphs, studied about trigonometric functions of sum and difference of real numbers, and deduced trigonometric functions of multiple and sub-multiples of real numbers.

In this lesson, we shall try to establish some results which will give the relationship between sides and angles of a triangle and will help in finding unknown parts of a triangle.



### OBJECTIVES

After studying this lesson, you will be able to :

- derive sine formula, cosine formula and projection formula
- apply these formulae to solve problems.

### EXPECTED BACKGROUND KNOWLEDGE

- Trigonometric functions.
- Formulae for sum and difference of trigonometric functions of real numbers.
- Trigonometric functions of multiples and sub-multiples of real numbers.

### 5.1 SINE FORMULA

In a  $\triangle ABC$ , the angles corresponding to the vertices A, B, and C are denoted by A, B, and C and the sides opposite to these vertices are denoted by a, b and c respectively. These angles and sides are called six elements of the triangle.

Prove that in any triangle, the lengths of the sides are proportional to the sines of the angles opposite to the sides,

i.e. 
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

**Proof :** In  $\triangle ABC$ , in Fig. 5.1 [(i), (ii) and (iii)],  $BC = a$ ,  $CA = b$  and  $AB = c$  and  $\angle C$  is acute angle in (i), right angle in (ii) and obtuse angle in (iii).

## MODULE - I

 Sets, Relations  
and Functions


Notes

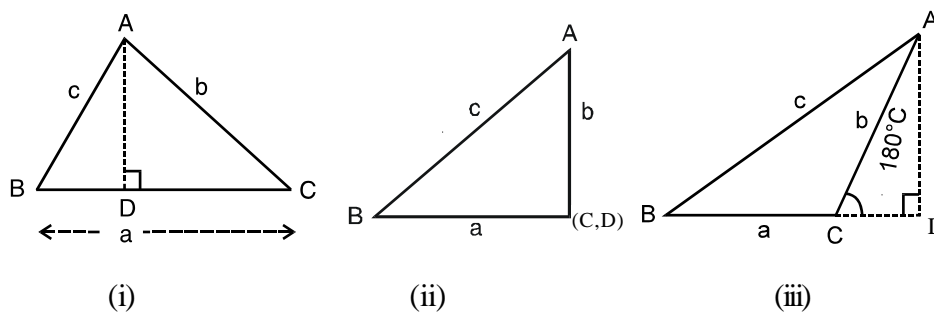


Fig. 5.1

Draw AD perpendicular to BC (or BC produced, if need be)

$$\text{In } \triangle ABC, \frac{AD}{AB} = \sin B \text{ or } \frac{AD}{c} = \sin B \Rightarrow AD = c \sin B \quad \dots(i)$$

$$\text{In } \triangle ADC, \frac{AD}{AC} = \sin C \text{ in Fig 5.1 (i)}$$

$$\text{or, } \frac{AD}{b} = \sin C \Rightarrow AD = b \sin C \quad \dots(ii)$$

$$\text{In Fig. 5.1 (ii), } \frac{AD}{AC} = 1 = \sin \frac{\pi}{2} = \sin C \text{ and } \frac{AD}{AB} = \sin B$$

$$AD = b \sin C \quad \text{and } AD = c \sin B.$$

$$\text{and in Fig. 5.1 (iii), } \frac{AD}{AC} = \sin(\pi - C) = \sin C \text{ and } \frac{AD}{AB} = \sin B$$

$$\text{or } \frac{AD}{b} = \sin C \text{ or } AD = b \sin C \text{ and } AD = c \sin B$$

$$\text{Thus, in all three figures, } AD = b \sin C \text{ and } AD = c \sin B \quad \dots(iii)$$

From (iii) we get

$$c \sin B = b \sin C \Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C} \quad \dots(iv)$$

Similarly, by drawing perpendiculars from C on AB, we can prove that

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \dots(v)$$

From (iv) and (v), we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \dots(A)$$

(A) is called the sine-formula

Note : (A) is sometimes written as



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \dots(A')$$

The relations (A) and (A') help us in finding unknown angles and sides, when some others are given.

Let us take some examples :

**Example 5.1** Prove that  $a \cos \frac{B-C}{2} = (b+c) \sin \frac{A}{2}$ , using sine-formula.

**Solution :** We know that,  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$  (say)

$$\Rightarrow a = k \sin A, b = k \sin B, c = k \sin C$$

$$\therefore \text{R.H.S.} = k(\sin B + \sin C) \cdot \sin \frac{A}{2}$$

$$= k \cdot 2 \sin \frac{B+C}{2} \cdot \cos \frac{B-C}{2} \cdot \sin \frac{A}{2}$$

Now  $\frac{B+C}{2} = 90^\circ - \frac{A}{2} \quad (\because A+B+C = \pi)$

$$\therefore \sin \frac{B+C}{2} = \cos \frac{A}{2}$$

$$\therefore \text{R.H.S.} = 2k \cos \frac{A}{2} \cdot \cos \frac{B-C}{2} \cdot \sin \frac{A}{2}$$

$$= k \cdot \sin A \cdot \cos \frac{B-C}{2} = a \cdot \cos \frac{B-C}{2} = \text{L.H.S}$$

**Example 5.2** Using sine formula, prove that

$$a(\cos C - \cos B) = 2(b-c) \cos^2 \frac{A}{2}$$

**Solution :** We have  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$  (say)

$$\Rightarrow a = k \sin A, b = k \sin B, c = k \sin C$$

$$\therefore \text{R.H.S} = 2k (\sin B - \sin C) \cdot \cos^2 \frac{A}{2}$$

$$= 2k \cdot 2 \cos \frac{B+C}{2} \cdot \sin \frac{B-C}{2} \cdot \cos^2 \frac{A}{2}$$

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$$\begin{aligned}
 &= 4k \sin \frac{A}{2} \cdot \sin \frac{B-C}{2} \cdot \cos^2 \frac{A}{2} = 2a \sin \frac{B-C}{2} \cdot \cos \frac{A}{2} \\
 &= 2a \sin \frac{B+C}{2} \cdot \sin \frac{B-C}{2} = a(\cos C - \cos B) = \text{L.H.S.}
 \end{aligned}$$

**Example 5.3** In any triangle ABC, show that

$$a \sin A - b \sin B = c \sin(A - B)$$

**Solution :** We have  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$  (say)

$$\text{L.H.S.} = k \sin A \cdot \sin A - k \sin B \cdot \sin B = k[\sin^2 A - \sin^2 B]$$

$$= k \sin(A + B) \cdot \sin(A - B)$$

$$A + B = \pi - C \Rightarrow \sin(A + B) = \sin C$$

$$\therefore \text{L.H.S.} = k \sin C \cdot \sin(A - B) = c \sin(A - B) = \text{R.H.S.}$$

**Example 5.4** In any triangle, show that

$$a(b \cos C - c \cos B) = b^2 - c^2$$

**Solution :** We have,  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$  (say)

$$\text{L.H.S.} = k \sin A (k \sin B \cos C - k \sin C \cos B) = k^2 \cdot \sin A [\sin(B - C)]$$

$$= k^2 \cdot \sin(B + C) \cdot \sin(B - C) \quad [\because \sin A = \sin(B + C)]$$

$$= k^2 (\sin^2 B - \sin^2 C) = k^2 \sin^2 B - k^2 \sin^2 C = b^2 - c^2 = \text{R.H.S.}$$


**CHECK YOUR PROGRESS 5.1**

1. Using sine-formula, show that each of the following hold :

$$(i) \quad \frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}} = \frac{a-b}{a+b} \qquad (ii) \quad b \cos B + c \cos C = a \cos(B - C)$$

$$(iii) \quad a \sin \frac{B-C}{2} = (b-c) \cos \frac{A}{2} \qquad (iv) \quad \frac{b+c}{b-c} = \tan \frac{B+C}{2} \cdot \cot \frac{B-C}{2}$$

$$(v) \quad a \cos A + b \cos B + c \cos C = 2a \sin B \sin C$$



2. In any triangle if  $\frac{a}{\cos A} = \frac{b}{\cos B}$ , prove that the triangle is isosceles.

### 5.2 COSINE FORMULA

In any triangle, prove that

$$(i) \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad (ii) \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac} \quad (iii) \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

**Proof :**

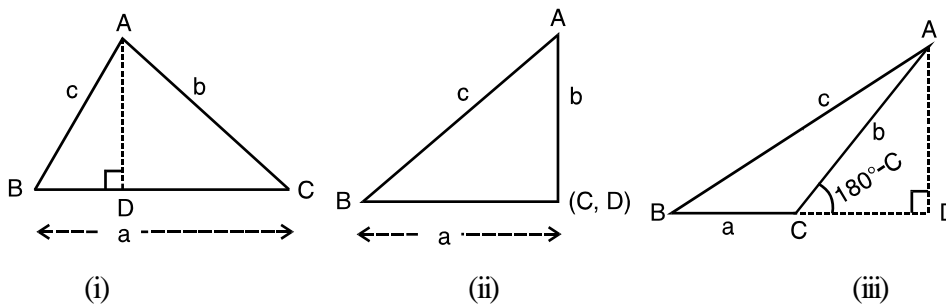


Fig. 5.2

Three cases arise :

- (i) When  $\angle C$  is acute
- (ii) When  $\angle C$  is a right angle
- (iii) When  $\angle C$  is obtuse

Let us consider these one by one :

**Case (i)** When  $\angle C$  is acute,  $\frac{AD}{AC} = \sin C \Rightarrow AD = b \sin C$

Also  $BD = BC - DC = a - b \cos C \quad \left[ \because \frac{DC}{b} = \cos C \right]$

From Fig. 5.2 (i)  $c^2 = (b \sin C)^2 + (a - b \cos C)^2$   
 $= b^2 \sin^2 C + a^2 + b^2 \cos^2 C - 2ab \cos C = a^2 + b^2 - 2ab \cos C$   
 $\Rightarrow \cos C = \frac{a^2 + b^2 - c^2}{2ab}$

**Case (ii)** When  $\angle C = 90^\circ$ ,  $c^2 = AD^2 + BD^2 = b^2 + a^2$

As  $C = 90^\circ \Rightarrow \cos C = 0 \therefore c^2 = b^2 + a^2 - 2ab \cdot \cos C$

$\Rightarrow \cos C = \frac{b^2 + a^2 - c^2}{2ab}$

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Case (iii) When  $\angle C$  is obtuse

$$\frac{AD}{AC} = \sin(180^\circ - C) = \sin C$$

$$\therefore AD = b \sin C$$

$$\text{Also, } BD = BC + CD = a + b \cos(180^\circ - C)$$

$$= a - b \cos C \quad \therefore c^2 = (b \sin C)^2 + (a - b \cos C)^2$$

$$= a^2 + b^2 - 2ab \cos C \Rightarrow \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\therefore \text{ In all the three cases, } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\text{Similarly, it can be proved that } \cos B = \frac{c^2 + a^2 - b^2}{2ac} \text{ and } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Let us take some examples to show its application.

**Example 5.5** In any triangle ABC, show that

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

**Solution :** We know that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{c^2 + a^2 - b^2}{2ac}, \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\therefore \text{ L.H.S.} = \frac{b^2 + c^2 - a^2}{2abc} + \frac{c^2 + a^2 - b^2}{2abc} + \frac{a^2 + b^2 - c^2}{2abc}$$

$$= \frac{1}{2abc} [b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2]$$

$$= \frac{a^2 + b^2 + c^2}{2abc} = \text{R.H.S.}$$

**Example 5.6** If  $\angle A = 60^\circ$ , show that in  $\triangle ABC$

$$(a + b + c)(b + c - a) = 3bc$$

$$\text{Solution : } \cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \dots(i) \quad \text{Here } A = 60^\circ \Rightarrow \cos A = \cos 60^\circ = \frac{1}{2}$$



$$\therefore \text{(i) becomes } \frac{1}{2} = \frac{b^2 + c^2 - a^2}{2bc} \quad \Rightarrow \quad b^2 + c^2 - a^2 = bc$$

$$\text{or } b^2 + c^2 + 2bc - a^2 = 3bc \quad \text{or } (b+c)^2 - a^2 = 3bc$$

$$\text{or } (b+c+a)(b+c-a) = 3bc$$

**Example 5.7** If the sides of a triangle are 3 cm, 5 cm and 7 cm find the greatest angle of the triangle.

**Solution :** Here  $a = 3$  cm,  $b = 5$  cm,  $c = 7$  cm

We know that in a triangle, the angle opposite to the largest side is greatest

$$\therefore \angle C \text{ is the greatest angle. } \therefore \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{9 + 25 - 49}{30} = \frac{-15}{30} = \frac{-1}{2}$$

$$\therefore \cos C = \frac{-1}{2} \quad \Rightarrow \quad C = \frac{2\pi}{3}$$

$\therefore$  The greatest angle of the triangle is  $\frac{2\pi}{3}$  or  $120^\circ$ .

**Example 5.8** In  $\triangle ABC$ , if  $\angle A = 60^\circ$ , prove that  $\frac{b}{c+a} + \frac{c}{a+b} = 1$ .

$$\text{Solution : } \cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{or} \quad \cos 60^\circ = \frac{1}{2} = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\therefore b^2 + c^2 - a^2 = bc \quad \text{or} \quad b^2 + c^2 = a^2 + bc \quad \dots\text{(i)}$$

$$\text{L.H.S.} = \frac{b}{c+a} + \frac{c}{a+b} = \frac{ab + b^2 + c^2 + ac}{(c+a)(a+b)}$$

$$= \frac{ab + ac + a^2 + bc}{(c+a)(a+b)} \quad [\text{Using (i)}]$$

$$= \frac{ab + a^2 + ac + bc}{(c+a)(a+b)} = \frac{a(a+b) + c(a+b)}{(a+c)(a+b)} = \frac{(a+c)(a+b)}{(a+c)(a+b)} = 1$$


**CHECK YOUR PROGRESS 5.2**

1. In any triangle ABC, show that

$$(i) \frac{b^2 - c^2}{a^2} \sin 2A + \frac{c^2 - a^2}{b^2} \sin 2B + \frac{a^2 - b^2}{c^2} \sin 2C = 0$$

$$(ii) (a^2 - b^2 + c^2) \tan B = (b^2 - c^2 + a^2) \tan C = (c^2 - a^2 + b^2) \tan A$$

$$(iii) \frac{k}{2} (\sin 2A + \sin 2B + \sin 2C) = \frac{a^2 + b^2 + c^2}{2abc}$$

$$\text{where } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$(iv) (b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$$

2. The sides of a triangle are  $a = 9$  cm,  $b = 8$  cm,  $c = 4$  cm. Show that  
 $6 \cos C = 4 + 3 \cos B$ .

**5.3 PROJECTION FORMULA**

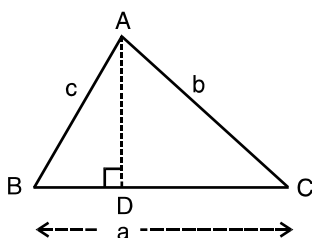
In  $\triangle ABC$ , if  $BC = a$ ,  $CA = b$  and  $AB = c$ , then prove that

$$(i) a = b \cos C + c \cos B$$

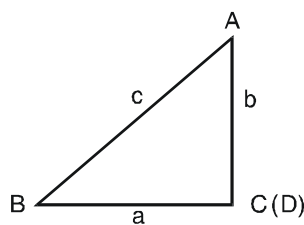
$$(ii) b = c \cos A + a \cos C$$

$$(iii) c = a \cos B + b \cos A$$

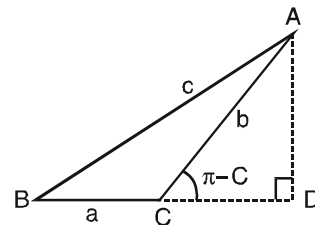
**Proof:**



(i)



(ii)



(iii)

**Fig. 5.3**

As in previous result, three cases arise. We will discuss them one by one.

(i) When  $\angle C$  is acute :

$$\text{In } \triangle ADB, \frac{BD}{c} = \cos B \quad \Rightarrow \quad BD = c \cos B$$

$$\text{In } \triangle ADC, \frac{DC}{b} = \cos C \quad \Rightarrow \quad DC = b \cos C$$

$$a = BD + DC = c \cos B + b \cos C, \therefore a = c \cos B + b \cos c$$

Notes





(ii) When  $\angle C = 90^\circ$

$$a = BC = \frac{BC}{AB} \cdot AB = \cos B \cdot c = c \cos B + 0$$

$$= c \cos B + b \cos 90^\circ \quad (\because \cos 90^\circ = 0) = c \cos B + b \cos C$$

(iii) When  $\angle C$  is obtuse

In  $\triangle ADB$ ,  $\frac{BD}{c} = \cos B \Rightarrow BD = c \cos B$

In  $\triangle ADC$ ,  $\frac{CD}{b} = \cos(\pi - C) = -\cos C \Rightarrow CD = -b \cos C$

In Fig. 5.3 (iii),

$$BC = BD - CD \Rightarrow a = c \cos B - (-b \cos C) = c \cos B + b \cos C$$

Thus in all cases,  $a = b \cos C + c \cos B$

Similarly, we can prove that

$$b = c \cos A + a \cos C \quad \text{and} \quad c = a \cos B + b \cos A$$

Let us take some examples, to show the application of these results.

**Example 5.9** In any triangle ABC, show that

$$(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$$

**Solution :** L.H.S. =  $b \cos A + c \cos A + c \cos B + a \cos B + a \cos C + b \cos C$

$$= (b \cos A + a \cos B) + (c \cos A + a \cos C) + (c \cos B + b \cos C)$$

$$= c + b + a = a + b + c = \text{R.H.S.}$$

**Example 5.10** In any  $\triangle ABC$ , prove that

$$\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

**Solution :** L.H.S. =  $\frac{1 - 2 \sin^2 A}{a^2} - \frac{1 - 2 \sin^2 B}{b^2}$

$$= \frac{1}{a^2} - \frac{2 \sin^2 A}{a^2} - \frac{1}{b^2} + \frac{2 \sin^2 B}{b^2}$$

$$= \frac{1}{a^2} - \frac{1}{b^2} - 2k^2 + 2k^2 = \frac{1}{a^2} - \frac{1}{b^2} \quad \left( \because \frac{\sin A}{a} = \frac{\sin B}{b} = k \right)$$

$$= \text{R. H. S.}$$

**Example 5.11** In  $\triangle ABC$ , if  $a \cos A = b \cos B$ , where  $a \neq b$  prove that  $\triangle ABC$  is a right angled triangle.

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$$\text{Solution : } a \cos A = b \cos B, \therefore a \left[ \frac{b^2 + c^2 - a^2}{2bc} \right] = b \left[ \frac{a^2 + c^2 - b^2}{2ac} \right]$$

$$\text{or } a^2 (b^2 + c^2 - a^2) = b^2 (a^2 + c^2 - b^2)$$

$$\text{or } a^2 b^2 + a^2 c^2 - a^4 = a^2 b^2 + b^2 c^2 - b^4$$

$$\text{or } c^2 (a^2 - b^2) = (a^2 - b^2) (a^2 + b^2)$$

$$\Rightarrow c^2 = a^2 + b^2 \quad \therefore \Delta ABC \text{ is a right triangle.}$$

**Example 5.12** If  $a = 2$ ,  $b = 3$ ,  $c = 4$ , find  $\cos A$ ,  $\cos B$  and  $\cos C$ .

$$\text{Solution : } \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{9 + 16 - 4}{2 \times 3 \times 4} = \frac{21}{24} = \frac{7}{8}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac} = \frac{16 + 4 - 9}{2 \times 4 \times 2} = \frac{11}{16}$$

$$\text{and } \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{4 + 9 - 16}{2 \times 2 \times 3} = \frac{-3}{12} = \frac{-1}{4}$$


**CHECK YOUR PROGRESS 5.3**

- If  $a = 3$ ,  $b = 4$  and  $c = 5$ , find  $\cos A$ ,  $\cos B$  and  $\cos C$ .
- The sides of a triangle are 7 cm,  $4\sqrt{3}$  cm and  $\sqrt{13}$  cm. Find the smallest angle of the triangle.
- If  $a : b : c = 7 : 8 : 9$ , prove that  $\cos A : \cos B : \cos C = 14 : 11 : 6$ .
- If the sides of a triangle are  $x^2 + x + 1$ ,  $2x + 1$  and  $x^2 - 1$ . Show that the greatest angle of the triangle is  $120^\circ$ .
- In a triangle,  $b \cos A = a \cos B$ , prove that the triangle is isosceles.
- Deduce sine formula from the projection formula.


**LET US SUM UP**

It is possible to find out the unknown elements of a triangle, if the relevant elements are given by using

**Sine-formula :**

$$(i) \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



**Cosine formulae :**

$$(ii) \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

**Projection formulae :**

$$a = b \cos C + c \cos B, \quad b = c \cos A + a \cos C, \quad c = a \cos B + b \cos A$$



**SUPPORTIVE WEB SITES**

[www.mathopenref.com/trianglesideangle.html](http://www.mathopenref.com/trianglesideangle.html)  
[http://en.wikipedia.org/wiki/Solution\\_of\\_triangles](http://en.wikipedia.org/wiki/Solution_of_triangles)  
[www.themathpage.com/abookI/propI-18-19.htm](http://www.themathpage.com/abookI/propI-18-19.htm)



**TERMINAL EXERCISE**

In a triangle ABC, prove the following (1-10) :

1.  $a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0$
2.  $a \cos A + b \cos B + c \cos C = 2a \sin B \sin C$
3.  $\frac{b^2 - c^2}{a^2} \cdot \sin 2A + \frac{c^2 - a^2}{b^2} \cdot \sin 2B + \frac{a^2 - b^2}{c^2} \cdot \sin 2C = 0$
4.  $\frac{c^2 + a^2}{b^2 + c^2} = \frac{1 + \cos B \cos (C - A)}{1 + \cos A \cos (B - C)}$
5.  $\frac{c - b \cos A}{b - c \cos A} = \frac{\cos B}{\cos C}$
6.  $\frac{a - b \cos C}{c - b \cos A} = \frac{\sin C}{\sin A}$
7.  $(a + b + c) \left[ \tan \frac{A}{2} + \tan \frac{B}{2} \right] = 2c \cot \frac{C}{2}$
8.  $\sin \frac{A - B}{2} = \frac{a - b}{c} \cos \frac{C}{2}$
9. (i)  $b \cos B + c \cos C = a \cos (B - C)$  (ii)  $a \cos A + b \cos B = c \cos (A - B)$
10.  $b^2 = (c - a)^2 \cos^2 \frac{B}{2} + (c + a)^2 \sin^2 \frac{B}{2}$
11. In a triangle, if  $b = 5, c = 6, \tan \frac{A}{2} = \frac{1}{\sqrt{2}}$ , then show that  $a = \sqrt{41}$ .
12. In any  $\Delta ABC$ , show that  $\frac{\cos A}{\cos B} = \frac{b - a \cos C}{a - b \cos C}$

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- $\cos A = \frac{4}{5}$   
 $\cos B = \frac{3}{5}$   
 $\cos C = \text{zero}$
- The smallest angle of the triangle is  $30^\circ$ .