9

QUADRATIC EQUATIONS AND LINEAR INEQUALITIES

Recall that an algebraic equation of the second degree is written in general form as $ax^2 + bx + c = 0,\ a \neq 0$. It is called a quadratic equation in x. The coefficient 'a' is the first or leading coefficient, 'b' is the second or middle coefficient and 'c' is the constant term (or third coefficient). For example, $7x^2 + 2x + 5 = 0$, $\frac{5}{2}x^2 + \frac{1}{2}x + 1 = 0$,

$3x^2 - x = 0$, $x^2 + \frac{1}{2} = 0$, $\sqrt{2}\,x^2 + 7x = 0$, are all quadratic equations.

Some times, it is not possible to translate a word problem in the form of an equation. Let us consider the following situation:

Alok goes to market with Rs. 30 to buy pencils. The cost of one pencil is Rs. 2.60. If x denotes the number of pencils which he buys, then he will spend an amount of Rs. 2.60x. This amount cannot be equal to Rs. 30 as x is a natural number. Thus.

$$2.60\,x < 30 \qquad \ldots \text{(i)}$$

Let us consider one more situation where a person wants to buy chairs and tables with Rs. 50,000 in hand. A table costs Rs. 550 while a chair costs Rs. 450. Let x be the number of chairs and y be the number of tables he buys, then his total cost = Rs.(550 x + 450 y)

Thus, in this case we can write, $550x + 450y \leq 50{,}000$

or $\quad 11x + 9y \leq 1000 \qquad \ldots \text{(ii)}$

Statement (i) involves the sign of inequality '<' and statement (ii) consists of two statements: $11x+9y < 1000$, $11x+9y = 1000$ in which the first one is not an equation: Such statements are called Inequalities. In this lesson, we will discuss linear inequalities and their solution.

We will also discuss how to solve quadratic equations with real and complex coefficients and establish relation between roots and coefficients.

OBJECTIVES

After studying this lesson, you will be able to:

- solve a quadratic equation with real coefficients by factorization and by using quadratic formula;

- find relationship between roots and coefficients;
- form a quadratic equation when roots are given;
- differentiate between a linear equation and a linear inequality;
- state that a planl region represents the solution of a linear inequality;
- represent graphically a linear inequality in two variables;
- show the solution of an inequality by shading the appropriate region;
- solve graphically a system of two or three linear inequalities in two variables;

EXPECTED BACKGROUND KNOWLEDGE

- Real numbers
- Quadratic Equations with real coefficients.
- Solution of linear equations in one or two variables.
- Graph of linear equations in one or two variables in a plane.
- Graphical solution of a system of linear equations in two variables.

9.1 ROOTS OF A QUADRATIC EQUATION

The value which when substituted for the variable in an equation, satisfies it, is called a root (or solution) of the equation.

If α be one of the roots of the quadratic equation

$$ax^2 + bx + c = 0,\ a \neq 0 \qquad \text{... (i)}$$

then $\quad a\alpha^2 + b\alpha + c = 0$

In other words, $x - \alpha$ is a factor of the quadratic equation (i)

In particular, consider a quadratic equation $x^2 + x - 6 = 0$ ...(ii)

If we substitute x = 2 in (ii), we get L.H.S $= 2^2 + 2 - 6 = 0$

$\therefore$ L.H.S = R.H.S.

Again put x = -3 in (ii), we get L.H.S. $= (-3)^2 - 3 - 6 = 0$

$\therefore$ L.H.S = R.H.S.

Again put x = -1 in (ii) ,we get L.H.S $= (-1)^2 + (-1) - 6 = -6 \neq 0$ = R.H.S.

$\therefore$ $x = 2$ and $x = -3$ are the only values of x which satisfy the quadratic equation (ii)

There are no other values which satisfy (ii)

$\therefore$ $x = 2,\ x = -3$ are the only two roots of the quadratic equation (ii)

Note: *If α, β be two roots of the quadratic equation*

$ax^2 + bx + c = 0,\ a \neq 0$ ...(A)

then $(x - \alpha)$ and $(x - \beta)$ will be the factors of (A). The given quadratic equation can be written in terms of these factors as $(x - \alpha)(x - \beta) = 0$

9.2 SOLVING QUADRATIC EQUATION BY FACTORIZATION

Recall that you have learnt how to factorize quadratic polynomial of the form $p(x) = ax^2 + bx + c,\ a \neq 0,$ by splitting the middle term and taking the common factors. Same method can be applied while solving a quadratic equation by factorization.

If $x - \frac{p}{q}$ and $x - \frac{r}{s}$ are two factors of the quadratic equation

$ax^2 + bx + c = 0, a \neq 0$ then $\left(x - \frac{p}{q}\right)\left(x - \frac{r}{s}\right) = 0$

$\therefore$ either $x = \frac{p}{q}$ or, $x = \frac{r}{s}$

$\therefore$ The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\frac{p}{q}, \frac{r}{s}$

Example 9.1 Using factorization method, solve the quadratic equation: $6x^2 + 5x - 6 = 0$

Solution: The given quadratic equation is $6x^2 + 5x - 6 = 0$ ... (i)

Splitting the middle term, we have $6x^2 + 9x - 4x - 6 = 0$

or, $3x(2x + 3) - 2(2x + 3) = 0$ or, $(2x + 3)(3x - 2) = 0$

$\therefore$ Either $2x + 3 = 0 \Rightarrow x = -\dfrac{3}{2}$ or, $3x - 2 = 0 \Rightarrow x = \dfrac{2}{3}$

$\therefore$ Two roots of the given quadratic equation are $-\dfrac{3}{2}, \dfrac{2}{3}$

Example 9.2 Using factorization method, solve the quadratic equation:

$$3\sqrt{2}x^2 + 7x - 3\sqrt{2} = 0$$

Solution: Splitting the middle term, we have $3\sqrt{2}x^2 + 9x - 2x - 3\sqrt{2} = 0$

or, $3x(\sqrt{2}x + 3) - \sqrt{2}(\sqrt{2}x + 3) = 0$ or, $(\sqrt{2}x + 3)(3x - \sqrt{2}) = 0$

$\therefore$ Either $\sqrt{2}x + 3 = 0 \Rightarrow x = -\dfrac{3}{\sqrt{2}}$ or, $3x - \sqrt{2} = 0 \Rightarrow x = \dfrac{\sqrt{2}}{3}$

$\therefore$ Two roots of the given quadratic equation are $-\dfrac{3}{\sqrt{2}}, \dfrac{\sqrt{2}}{3}$

Example 9.3 Using factorization method, solve the quadratic equation:

$$(a + b)^2 x^2 + 6(a^2 - b^2)x + 9(a - b)^2 = 0$$

Solution: The given quadratic equation is $(a + b)^2 x^2 + 6(a^2 - b^2)x + 9(a - b)^2 = 0$

Splitting the middle term, we have

$(a + b)^2 x^2 + 3(a^2 - b^2)x + 3(a^2 - b^2)x + 9(a - b)^2 = 0$

or, $(a + b)x\{(a + b)x + 3(a - b)\} + 3(a - b)\{(a + b)x + 3(a - b)\} = 0$

or, $\{(a + b)x + 3(a - b)\}\{(a + b)x + 3(a - b)\} = 0$

$\therefore$ either $(a + b)\,x + 3\,(a - b) = 0 \Rightarrow x = \dfrac{-3(a-b)}{a+b} = \dfrac{3\,(b-a)}{a+b}$

or, $(a + b)\,x + 3\,(a - b) = 0 \Rightarrow x = \dfrac{-3(a-b)}{a+b} = \dfrac{3\,(b-a)}{a+b}$

The equal roots of the given quadratic equation are $\dfrac{3\,(b-a)}{a+b}, \dfrac{3\,(b-a)}{a+b}$

<u>Alternative Method</u>

The given quadratic equation is $(a + b)^2 x^2 + 6(a^2 - b^2)\,x + 9(a - b)^2 = 0$

This can be rewritten as

$\{(a + b)\,x\}^2 + 2\,.(a + b)x\,.\,3\,(a - b) + \{3(a - b)\}^2 = 0$

or, $\{\,(a + b)x + 3(a - b)\,\}^2 = 0$ or, $x = -\dfrac{3(a-b)}{a+b} = \dfrac{3\,(b-a)}{a+b}$

$\therefore$ The quadratic equation has equal roots $\dfrac{3\,(b-a)}{a+b}, \dfrac{3\,(b-a)}{a+b}$

CHECK YOUR PROGRESS 9.1

1. Solve each of the following quadratic equations by factorization method:

(i) $\sqrt{3}\,x^2 + 10x + 8\sqrt{3} = 0$ (ii) $x^2 - 2ax + a^2 - b = 0$

(iii) $x^2 + \left(\dfrac{ab}{c} - \dfrac{c}{ab}\right) x - 1 = 0$ (iv) $x^2 - 4\sqrt{2}\,x + 6 = 0$

9.3 SOLVING QUADRATIC EQUATION BY QUADRATIC FORMULA

Recall the solution of a standard quadratic equation

$ax^2 + bx + c = 0,\ a \neq 0$ by the **"Method of Completing Squares"**

Roots of the above quadratic equation are given by

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{D}}{2a}, \qquad\qquad = \frac{-b-\sqrt{D}}{2a}$$

where $D = b^2 - 4ac$ is called the discriminant of the quadratic equation.

For a quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$ if

(i) **D>0, the equation will have two real and unequal roots**

(ii) **D=0, the equation will have two real and equal roots and both roots are equal to $-\dfrac{b}{2a}$**

(iii) **D<0, the equation will have two conjugate complex (imaginary) roots.**

Example 9.4 Examine the nature of roots in each of the following quadratic equations and also verify them by formula.

(i) $x^2 + 9x + 10 = 0$ (ii) $9y^2 - 6\sqrt{2}y + 2 = 0$

(iii) $\sqrt{2}t^2 - 3t + 3\sqrt{2} = 0$

Solution:

(i) The given quadratic equation is $x^2 + 9x + 10 = 0$

Here, $a = 1$, $b = 9$ and $c = 10$

$\therefore$ $D = b^2 - 4ac = 81 - 4.1.10 = 41 > 0.$

$\therefore$ The equation will have two real and unequal roots

Verification: By quadratic formula, we have $x = \dfrac{-9 \pm \sqrt{41}}{2}$

$\therefore$ The two roots are $\dfrac{-9+\sqrt{41}}{2}, \dfrac{-9-\sqrt{41}}{2}$ which are real and unequal.

(ii) The given quadratic equation is $9y^2 - 6\sqrt{2}y + 2 = 0$

Here, $D = b^2 - 4ac = (-6\sqrt{2})^2 - 4.9.2 = 72 - 72 = 0$

$\therefore$ The equation will have two real and equal roots.

Verification: By quadratic formula, we have $y = \dfrac{6\sqrt{2} \pm \sqrt{0}}{2.9} = \dfrac{\sqrt{2}}{3}$

$\therefore$ The two equal roots are $\dfrac{\sqrt{2}}{3}, \dfrac{\sqrt{2}}{3}$.

(iii) The given quadratic equation is $\sqrt{2}t^2 - 3t + 3\sqrt{2} = 0$

Here, $D = (-3)^2 - 4.\sqrt{2}.3\sqrt{2} = -15 < 0$

$\therefore$ The equation will have two conjugate complex roots.

MODULE-III
Algebra -I

Notes

Verification: By quadratic formula, we have $t = \dfrac{3 \pm \sqrt{-15}}{2\sqrt{2}} = \dfrac{3 \pm \sqrt{15}i}{2\sqrt{2}}$, where $i = \sqrt{-1}$

$\therefore$ Two conjugate complex roots are $\dfrac{3+\sqrt{15}\,i}{2\sqrt{2}}, \dfrac{3-\sqrt{15}\,i}{2\sqrt{2}}$

Example 9.5 Prove that the quadratic equation $x^2 + px - 1 = 0$ has real and distinct roots for all real values of p.

Solution: Here, $D = p^2 + 4$ which is always positive for all real values of p.

$\therefore$ The quadratic equation will have real and distinct roots for all real values of p.

Example 9.6 For what values of k the quadratic equation

$(4k+ 1)\, x^2 + (k + 1)\, x + 1 = 0$ will have equal roots ?

Solution: The given quadratic equation is $(4k + 1)x^2 + (k + 1)\, x + 1 = 0$

Here, $D = (k + 1)^2 - 4.(4k + 1).1$

For equal roots, $D = 0 \;\therefore\; (k + 1)^2 - 4\,(4k + 1) = 0$

$\Rightarrow \quad k^2 - 14k - 3 = 0$

$\therefore \quad k = \dfrac{14 \pm \sqrt{196+12}}{2}$ or $k = \dfrac{14 \pm \sqrt{208}}{2} = 7 \pm 2\sqrt{13}$ or $7 + 2\sqrt{13}, 7 - 2\sqrt{13}$

which are the required values of k.

Example 9.7 Prove that the roots of the equation

$x^2\,(a^2+ b^2) + 2x\,(ac+ bd) + (c^2+ d^2) = 0$ are imaginary. But if $ad = bc$, roots are real and equal.

Solution: The given equation is $x^2 (a^2 + b^2) + 2x\,(ac + bd) + (c^2 + d^2) = 0$

Discriminant $= 4\,(ac + bd)^2 - 4\,(a^2 + b^2)\,(c^2 + d^2)$

$= 8\,abcd - 4(a^2 d^2 + b^2 c^2) = -4\,(-2\,abcd + a^2 d^2 + b^2 c^2)$

$= -4\,(ad - bc)^2, <0$ for all a, b, c, d

$\therefore$ The roots of the given equation are imaginary.

For real and equal roots, discriminant is equal to zero.

$\Rightarrow \quad -4\,(ad - bc)^2 = 0$ or, $ad = bc$

Hence, if ad=bc, the roots are real and equal.

CHECK YOUR PROGRESS 9.2

1. Solve each of the following quadratic equations by quadratic formula:

(i) $2x^2 - 3x + 3 = 0$ (ii) $-x^2 + \sqrt{2}x - 1 = 0$

(iii) $-4x^2+\sqrt{5}x-3=0$ (iv) $3x^2+\sqrt{2}x+5=0$

2. For what values of k will the equation

   $y^2-2(1+2k)y+3+2k=0$ have equal roots ?

3. Show that the roots of the equation

   $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$ are always real and they can not be equal unless $a=b=c$.

9.4 RELATION BETWEEN ROOTS AND COEFFICIENTS OF A QUADRATIC EQUATION

You have learnt that, the roots of a quadratic equation $ax^2+bx+c=0,\ a\neq 0$

are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$

Let $\alpha=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ ...(i) and $\beta=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$ ... (ii)

Adding (i) and (ii), we have $\alpha+\beta=\dfrac{-2b}{2a}=\dfrac{-b}{a}$

$\therefore$ **Sum of the roots** $= -\dfrac{\textit{coefficient of } x}{\textit{coefficient of } x^2}=-\dfrac{b}{a}$ ... (iii)

$$\alpha\ \beta=\frac{+b^2-(b^2-4ac)}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}$$

$\therefore$ **Product of the roots** $=\dfrac{\text{constant term}}{\text{coefficient of } x^2}=\dfrac{c}{a}$ ...(iv)

(iii) and (iv) are the required relationships between roots and coefficients of a given quadratic equation. These relationships helps to find out a quadratic equation when two roots are given.

Example 9.8 If, α,β are the roots of the equation $3x^2-5x+9=0$ find the value of:

(a) $\alpha^2+\beta^2$ (b) $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$

Solution: (a) It is given that α,β are the roots of the quadratic equation $3x^2-5x+9=0$.

$\therefore$ $\alpha+\beta=\dfrac{5}{3}$ ... (i)

and $\alpha\beta=\dfrac{9}{3}=3$ ... (ii)

Now, $\alpha^2+\beta^2 \quad =(\alpha+\beta)^2-2\alpha\beta \quad = \left(\frac{5}{3}\right)^2 - 2.3$ [By (i) and (ii)]

$$= -\frac{29}{9}$$

(b) Now, $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}=\frac{\frac{-29}{9}}{9}$ [By (i) and (ii)]

$$= -\frac{29}{81}$$

Example 9.9 If α, β are the roots of the equation $3y^2 + 4y + 1 = 0$, form a quadratic equation whose roots are α^2, β^2

Solution: It is given that α, β are two roots of the quadratic equation $3y^2 + 4y + 1 = 0$.

$\therefore$ Sum of the roots

i.e., $\alpha + \beta = -\frac{\text{coefficient of } y}{\text{coefficient of } y^2} = -\frac{4}{3}$ ... (i)

Product of the roots i.e., $\alpha\ \beta = \frac{\text{constant term}}{\text{coefficient of } y^2} = \frac{1}{3}$ ... (ii)

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(-\frac{4}{3}\right)^2 - 2.\frac{1}{3}$ [By (i) and (ii)]

$= \frac{16}{9} - \frac{2}{3} = \frac{10}{9}$

and $\alpha^2\beta^2 = (\alpha\beta)^2 = \frac{1}{9}$ [By (i)]

$\therefore$ The required quadratic equation is $y^2 - (\alpha^2 + \beta^2)y + \alpha^2\beta^2 = 0$

or, $y^2 - \frac{10}{9}y + \frac{1}{9} = 0$ or, $9y^2 - 10y + 1 = 0$

Example 9.10 If one root of the equation $ax^2+bx+c=0,\ a\neq 0$ be the square of the other, prove that $b^3 + ac^2 + a^2c = 3abc$

Solution: Let α, α^2 be two roots of the equation $ax^2 + bx + c = 0$.

$\therefore \qquad \alpha + \alpha^2 = -\frac{b}{a}$ ... (i)

and $\qquad \alpha . \alpha^2 = \frac{c}{a}$

i.e., $\qquad \alpha^3 = \frac{c}{a}$ . ... (ii)

From (i) we have $\alpha(\alpha+1) = -\frac{b}{a}$

or, $\qquad \{\alpha(\alpha+1)\}^3 = \left(-\frac{b}{a}\right)^3 = -\frac{b^3}{a^3}$ or, $\alpha^3(\alpha^3 + 3\alpha^2 + 3\alpha + 1) = -\frac{b^3}{a^3}$

or, $\qquad \frac{c}{a}\left\{\frac{c}{a} + 3\left(-\frac{b}{a}\right) + 1\right\} = -\frac{b^3}{a^3}$ ... [By (i) and (ii)]

or, $\qquad \frac{c^2}{a^2} - \frac{3bc}{a^2} + \frac{c}{a} = -\frac{b^3}{a^3}$ or, $ac^2 - 3abc + a^2c = -b^3$

or, $b^3 + ac^2 + a^2c = 3abc$, which is the required result.

Example 9.11 Find the condition that the roots of the equation $ax^2 + bx + c = 0$ are in the ratio m : n

Solution: Let $m\alpha$ and $n\alpha$ be the roots of the equation $ax^2 + bx + c = 0$

Now, $\qquad m\alpha + n\alpha = -\frac{b}{a}$ ... (i)

and $\qquad mn\alpha^2 = \frac{c}{a}$ ... (ii)

From (i) we have, $\qquad \alpha(m+n) = -\frac{b}{a}$ or, $\alpha^2(m+n)^2 = \frac{b^2}{a^2}$

or, $\qquad \frac{c}{a}(m+n)^2 = mn\frac{b^2}{a^2}$ [By (ii)]

or, $\qquad ac(m+n)^2 = mn\,b^2$, which is the required condition

CHECK YOUR PROGRESS 9.3

1. If α, β are the roots of the equation $ay^2 + by + c = 0$ then find the value of :

(i) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ (ii) $\frac{1}{\alpha^4}+\frac{1}{\beta^4}$

2. If α, β are the roots of the equation $5x^2 - 6x + 3 = 0$, form a quadratic equation whose roots are:

(i) α^2, β^2 (ii) $\alpha^3\beta, \alpha\beta^3$

3. If the roots of the equation $ay^2 + by + c = 0$ be in the ratio 3:4, prove that $12b^2 = 49\,ac$

4. Find the condition that one root of the quadratic equation $px^2 - qx + p = 0$ may be 1 more than the other.

9.5 SOLUTION OF A QUADRATIC EQUATION WHEN D < 0

Let us consider the following quadratic equation:

(a) Solve for t : $t^2 + 3t + 4 = 0$

$$\therefore \quad t = \frac{-3 \pm \sqrt{9-16}}{2} = \frac{-3 \pm \sqrt{-7}}{2}$$

Here, $D = -7 < 0$

$$\therefore \quad \text{The roots are } \frac{-3+\sqrt{-7}}{2} \text{ and } \frac{-3-\sqrt{-7}}{2}$$

or, $$\frac{-3+\sqrt{7}\,i}{2}, \frac{-3-\sqrt{7}\,i}{2}$$

Thus, the roots are complex and conjugate.

(b) Solve for y :

$$-3y^2 + \sqrt{5}\,y - 2 = 0$$

$$\therefore \quad y = \frac{-\sqrt{5} \pm \sqrt{5-4(-3).(-2)}}{2(-3)} \text{ or } y = \frac{-\sqrt{5} \pm \sqrt{-19}}{-6}$$

Here, $D = -19 < 0$

$$\therefore \quad \text{The roots are } \frac{-\sqrt{5}+\sqrt{19}i}{-6}, -\frac{\sqrt{5}-\sqrt{19}i}{-6}$$

Here, also roots are complex and conjugate. From the above examples , we can make the following conclusions:

(i) $D < 0$ in both the cases

(ii) Roots are complex and conjugate to each other.

Is it always true that complex roots occur in conjugate pairs ?

Let us form a quadratic equation whose roots are $2 + 3i$ and $4 - 5i$

The equation will be $\{x-(2+3i)\}\ \{x-(4-5i)\}=0$

or, $x^2-(2+3i)x-(4-5i)x+(2+3i)(4-5i)=0$

or, $x^2+(-6+2i)x+23+2i=0$, which is an equation with complex coefficients.

> ***Note*** **:** *If the quadratic equation has two complex roots, which are not conjugate of each other, the quadratic equation is an equation with complex coefficients.*

9.6 Fundamental Theorem of Algebra

You may be interested to know as to how many roots does an equation have? In this regard the following theorem known as fundamental theorem of algebra, is stated (without proof). 'A polynomial equation has at least one root'.

As a consequence of this theorem, the following result, which is of immense importance is arrived at.

'A polynomial equation of degree *n* has exactly *n* roots'

CHECK YOUR PROGRESS 9.4

Solve each of the following equations.

1. $-x^2+x+2=0$
2. $\sqrt{3}x^2-\sqrt{2}x+3\sqrt{3}=0$
3. $x^2+\dfrac{1}{\sqrt{2}}x+1=0$

4. $\sqrt{5}x^2+x+\sqrt{5}=0$
4. $x^2+3x+5=0$

9.7 INEQUALITIES (INEQUATIONS)

Now we will discuss about linear inequalities and their applications from daily life. A statement involving a sign of equality (=) is an equation.

Similarly, a statement involving a sign of inequality, $<, >, \leq$, or $\geq$ is called an inequalities.

Some examples of inequalities are:

(i) $2x+5>0$ (ii) $3x-7<0$

(iii) $ax+b\geq 0$, $a\neq 0$ (iv) $ax+b\leq c$, $a\neq 0$

(v) $3x+4y\leq 12$ (vi) $x^2-5x+6<0$

(vii) $ax+by+c\geq 0$

(v) and (vii) are inequalities in two variables and all other inequalities are in one variable. (i) to (v) and (vii) are linear inequalities and (vi) is a quadratic inequalities.

In this lesson, we shall study about linear inequalities in one or two variables only.

9.8 SOLUTIONS OF LINEAR INEQUALITIES IN ONE/TWO VARIABLES

Solving an inequalities means to find the value (or values) of the variable (s), which when substituted in the inequalities, satisfies it.

For example, for the inequalities $2.60x<30$ (statement) (i) all values of $x \leq 11$ are the solutions. (x is a whole number)

For the inequalities $2x + 16>0$, where x is a real number, all values of x which are > -8 are the solutions.

For the linear inequation in two variables, like $ax + by + c \geq 0$, we shall have to find the pairs of values of x and y which make the given inequalities true.

Let us consider the following situation :

Anil has Rs. 60 and wants to buy pens and pencils from a shop. The cost of a pen is Rs. 5 and that of a pencil is Rs. 3 If x denotes the number of pens and y, the number of pencils which Anil buys, then we have the inequality $5x + 3y \leq 60$ ... (i)

Here, $x = 6$, $y = 10$ is one of the solutions of the inequalities (i). Similarly $x = 5$, $y = 11$; $x = 4$, $y = 13$; $x = 10$, $y = 3$ are some more solutions of the inequalities.

In solving inequalities, we follow the rules which are as follows :

1. Equal numbers may be added (or subtracted) from both sides of an inequalities.

   Thus (i) if $a > b$ then $a + c > b + c$ and $a - c > b - c$

   and (ii) if $a \leq b$ then $a + d \leq b + d$ and $a - d \leq b - d$

2. Both sides of an inequalities can be multiplied (or divided) by the same positive number.

   Thus (i) if $a > b$ and $c > 0$ then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$

   and (ii) if $a \leq b$ and c > 0 then $ac \leq bc$ and $\frac{a}{c} \leq \frac{b}{c}$

3. When both sides of an inequalities are multiplied by the same negative number, the sign of inequality gets reversed.

   Thus (i) if $a > b$ and $d < 0$ then $ad < bd$ and $\frac{a}{d} < \frac{b}{d}$

   and (ii) if $a \leq b$ and $c < 0$ then $ac \geq bc$ and $\frac{a}{c} \geq \frac{b}{c}$

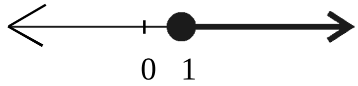
Example 9.12 Solve $\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$. Show the graph of the solutions on number line.

Solution: We have

$$\frac{3x-4}{2} \geq \frac{x+1}{4} - 1 \text{ or } \frac{3x-4}{2} \geq \frac{x+3}{4}$$

$$\text{or} \quad 2(3x-4) \geq (x-3) \text{ or } 6x-8 \geq x-3 \text{ or } 5x \geq 5 \; or \; x \geq 1$$

The graphical representation of solutions is given in Fig.



Example 9.13 The marks obtained by a student of Class XI in first and second terminal examination are 62 and 48, respectively. Find the minimum marks he should get in the annual examination to have an average of at least 60 marks.

Solution: Let x be the marks obtained by student in the annual examination. Then

$$\frac{62+48+x}{3} \geq 60 \text{ or } 110+x \geq 180 \text{ or } x \geq 70$$

Thus, the student must obtain a minimum of 70 marks to get an average of at least 60 marks.

Example 9.14 A manufacturer has 600 litres of a 12% solution of acid. How many litres of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18%?

Solution: Let x litres of 30% acid solution is required to be added. Then

Total mixture = (x + 600) litres

Therefore $\quad 30\%\ x + 12\% \text{ of } 600 > 15\% \text{ of } (x + 600)$

and $\quad 30\%\ x + 12\% \text{ of } 600 < 18\% \text{ of } (x + 600)$

or $$\frac{30x}{100}+\frac{12}{100}(600) > \frac{15}{100}(x+600)$$

and $$\frac{30x}{100}+\frac{12}{100}(600) < \frac{18}{100}(x+600)$$

or $\quad 30x + 7200 > 15x + 9000$

and $\quad 30x + 7200 < 18x + 10800$

or $\quad 15x > 1800$ and $12x < 3600$

or $\quad x > 120$ and $x < 300$,

i.e. $\quad 120 < x < 300$

Thus, the number of litres of the 30% solution of acid will have to be more than 120 litres but less than 300 litres.

6.3 GRAPHICAL REPRESENTATION OF LINEAR INEQUALITIES IN ONE OR TWO VARIABLES.

In Section 6.2, while translating word problem of purchasing pens and pencils, we obtained the following linear inequalities in two variables x and y:

$5x + 3y \leq 60$ .... (i)

Let us now find all solutions of this inequation, keeping in mind that x and y here can be only whole numbers.

To start with, let $x = 0$.

Thus, we have $3y \leq 60$ or $y \leq 20$, i,e the values of y corresponding to $x = 0$ can be 0,1,2,3............, 20 only Thus, the solutions with $x = 0$ are

(0,0), (0,1), (0,2) ....(0, 20)

Similarly the other solutions of the inequalities, when $x = 1,2, \ldots 12$ are

(1,0) (1,1) (1,2) .... (1,18)

(2,0) (2,1) (2,2) .... (2,16)

....

....

(10,0) (10,1) (10,2), (10,3)

(11,0) (11,1)

(12,0)

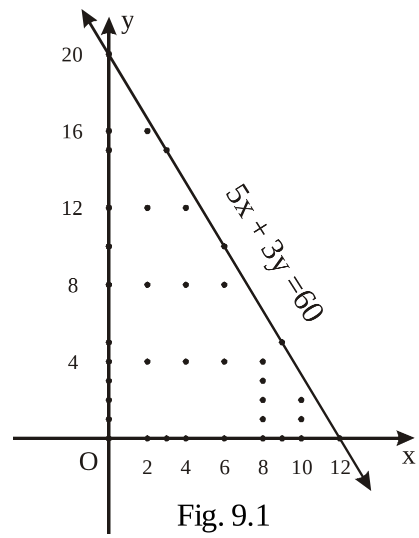


Fig. 9.1

You may note that out of the above ordered pairs, some pairs such as (0,20), (3, 15), (6, 10), (9, 5), (12,0) satisfy the equation $5x + 3y = 60$ which is a part of the given inequation and all other possible solutions lie on **<u>one of the two half planes</u>** in which the line $5x + 3y = 60$, divides the xy - plane.

If we now extend the domain of x and y from whole numbers to real numbers, the inequation $5x + 3y \leq 60$ will represent one of the two half planes in which the line $5x + 3y = 60$, divides the xy-plane.

Thus we can generalize as follows :

If a,b,c, are real numbers, then $ax + by + c = 0$ is called a linear equalities in two variables x and y, where as $ax + by + c \leq 0$ or $ax + by + c \geq 0$, $ax + by + c > 0$ and $ax + by + c < 0$ are called linear inequations in two variables x and y.

The equation $ax + by + c = 0$ is a straight line which divides the xy plane into two half planes which are represented by $ax + by + c \geq 0$ and $ax + by + c \leq 0$.

For example $3x + 4y - 12 = 0$ can be represented by line AB, in the xy - plane as shown in Fig. 9.2

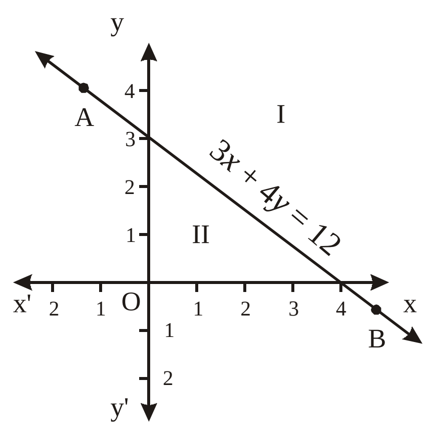


Fig. 9.2

The line AB divides the cordinate plane into two half -plane regions :

(i) half plane region I above the line AB

(ii) half plane region II below the line AB. One of the above region represents the inequality $3x + 4y - 12 \le 0$ ...(i) and the other region will be represented by $3x + 4y - 12 \ge 0$. ... (ii)

To identify the half plane represented by inequation (i), we take any arbitrary point, preferably origin, if it does not lie on AB. If the point satisfies the inequation (i), then the half plane in which the arbitrary point lies, is the desired half plane. In this case, taking origin as the arbitrary point we have

$0+0-12 \le 0$ i.e $-12 \le 0$. Thus origin satisfies the inequalities $3x + 4y - 12 \le 0$. Now, origin lies in half plane region II. Hence the inequalty $3x + 4y - 12 \le 0$ represents half plane II and the inequality $3x + 4y - 12 \ge 0$ will represent the half plane I

Example 9.15 Show on graph the region represented by the inequalities $x + 2y \ge 5$.

Solution : The given inequalities is $x + 2y \ge 5$

Let us first take the corresponding linear equation $x + 2y = 5$ and draw its graph with the help of the following table :

x	1	3	5
y	2	1	0

Since (0,0) does not lie on the line AB, so we can select (0,0) as the arbitrary point. Since $0 + 0 \ge 5$ is not true

$\therefore$ The desired half plane is one, in which origin does not lie

$\therefore$ The desired half plane is the shaded one (See Fig. 9.3)

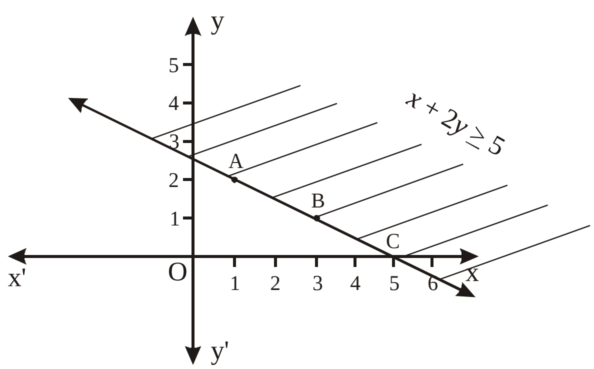


Fig. 9.3

Before taking more examples, it is important to define the following :

(i) Closed Half Plane: A half plane is said to be closed half plane if all points on the line separating the two half planes are also included in the solution of the inequation. The Half plane in Example 6.1 is a closed half plane.

(ii) An Open Half Plane : A half plane in the xy plane is said to be an open half plane if the points on the line separting the planes are not included in the half plane.

Example 9.16 Draw the graph of inequation $x - 5y > 0$

Solution : The given inequation is $x - 5y > 0$

The corresponding linear equation is $x - 5y = 0$ we have the following table.

x	0	5	–5
y	0	1	–1

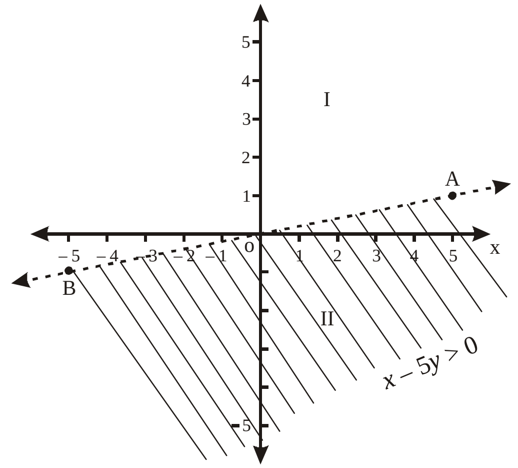


Fig. 9.4

The line AOB divides xy - plane into two half planes I and II. As the line AOB passes through origin, we consider any other arbitrary point (say) P (3,4) which is in half plane I. Let us see whether it satisfies the given inequation $x - 5y > 0$

$\therefore$ Then $3 - 5(4) > 0$ or $3 - 20 > 0$, or $-17 > 0$ which is not true

$\therefore$ The desired half plane is II

Again the inequation is a **strict** inequation $x - 5y > 0$

$\therefore$ Line AOB is not a part of the graph and hence has been shown as a dotted line.

Hence, the graph of the given inequation is the shaded region half plane II excluding the line AOB.

Example 9.17 Represent graphically the inequlities $3x - 12 \geq 0$

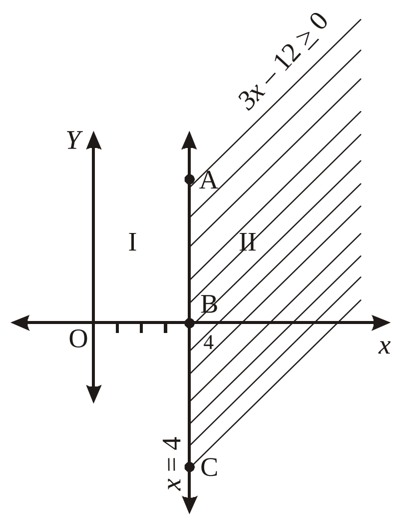


Fig. 9.5

Solution : Given inequation is $3x - 12 \geq 0$ and the corresponding linear equation is $3x - 12 = 0$ or $x - 4 = 0$ or $x = 4$ which is represented by the line ABC on the *xy* - plane (See Fig. 9.5). Taking (0,0) as the arbitrary point, we can say that $0 \neq 4$ and so, half plane II represents the inequation $3x - 12 \geq 0$

Example 9.18 Solve graphically the inequation $2y + 4 \geq 0$

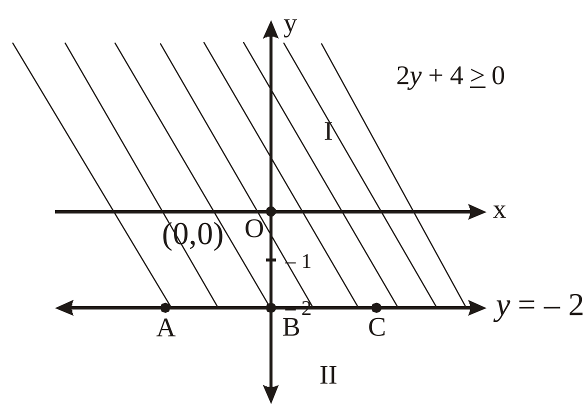


Fig. 9.6

Solution : Here the inequation is

$2y + 4 \geq 0$ and the corresponding equation is $2y + 4 = 0$ or $y = -2$

The line ABC represents the line
$y = -2$ which divides the xy - plane into two half planes and the inequation
$2y + 4 \geq 0$ is represented by the half plane I.

CHECK YOUR PROGRESS 9.5

Represent the solution of each of the following inequations graphically in two dimensional plane:

1. $2x + y \geq 8$ 2. $x - 2y \leq 0$
3. $3x + 6 \geq 0$ 4. $8 - 2y \geq 2$
5. $3y \geq 6 - 2x$ 6. $3x \geq 0$
7. $y \leq 4$ 8. $y > 2x - 8$
9. $-y < x - 5$ 10. $2y \leq 8 - 4x$

6.4 GRAPHICAL SOLUTION OF A SYSTEM OF LINEAR INEQUATIONS IN TWO VARIABLES.

You already know how to solve a system of linear equations in two variables.

Now, you have also learnt how to solve **linear inequations** in two variables graphically. We will now discuss the technique of finding the solutions of a system of simultaneous linear inequations. By the term solution of a system of simultaneons linear inequations we mean, finding all ordered pairs (x,y) for which each linear inequation of the system is satisfied.

A system of simultaneous inequations may have no solution or an infinite number of solutions represented by the region bounded or unbounded by straight lines corresponding to linear inequations.

We take the following example to explain the technique.

Example 9.19 Solve the following system of inequations graphically:

$$x + y \geq 6 ; \qquad 2x - y \geq 0.$$

Solution : Given inequations are

$$x + y \geq 6 \text{ ..... (i)}$$

and $\quad 2x - y \geq 0$ ..... (ii)

We draw the graphs of the lines $x + y = 6$ and $2x - y = 0$ (Fig. 9.7)

The inequation (i) represent the shaded region above the line $x + y = 6$ and inequations (ii) represents the region on the right of the line $2x - y = 0$

The common region represented by the double shade in Fig. 9.7 represents the solution of the given system of linear inequations.

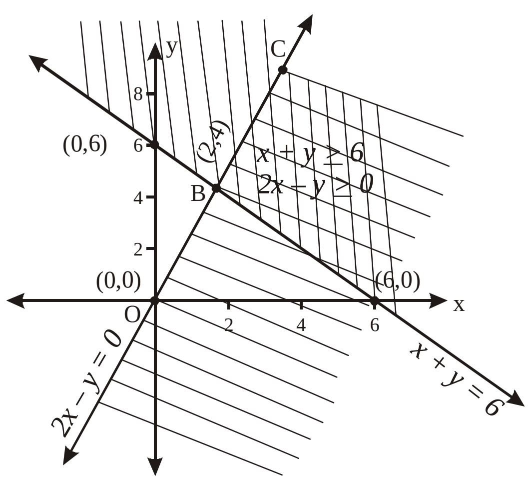


Fig. 9.7

Example 9.20 Find graphically the solution of the following system of linear inequations :

$x + y \leq 5, \qquad 4x + y \geq 4,$
$x + 5y \geq 5, \qquad x \leq 4, \;\; y \leq 3.$

Solution : Given inequations are

$x + y \leq 5$ ..... (i)
$4x + y \geq 4$ ..... (ii)
$x + 5y \geq 5$ ..... (iii)
$x \leq 4$ ..... (iv)
and $y \leq 3$ ..... (v)

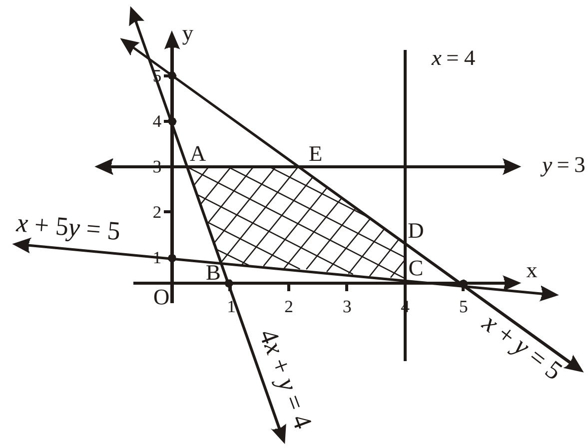


Fig. 9.8

We draw the graphs of the lines
$x + y = 5, \;\; 4x + y = 4, \;\; x + 5y = 5,$
$x = 4$ and $y = 3$ (Fig. 9.8)

The inequlities (i) represents the region below the line $x + y = 5$. The inequations (ii) represents the region on the right of equation $4x + y = 4$ and the region above the line $x + 5y = 5$ represents the inequation (iii). Similarly after shading the regions for inequations (iv) and (v) we get the common region as the bounded region ABCDE as shown in (Fig. 9.8) The co-ordinates of the points of the shaded region satisfy the given system of inequations and therefore all these points represent solution of the given system.

Example 9.21 Solve graphically the following system of inequations :

$x + 2y \leq 3, 3x + 4y \geq 12, \;\; x \geq 0, y \geq 0.$

Solution : We represent the inequations $x + 2y \leq 3$, $3x + 4y \geq 12$, $x \geq 0$, $y \geq 0$ by shading the corresponding regions on the graph as shown in Fig. 9.9

Here we find that there is no common region represented by these inequations.

We thus conclude that there is no solution of the given system of linear inequations.

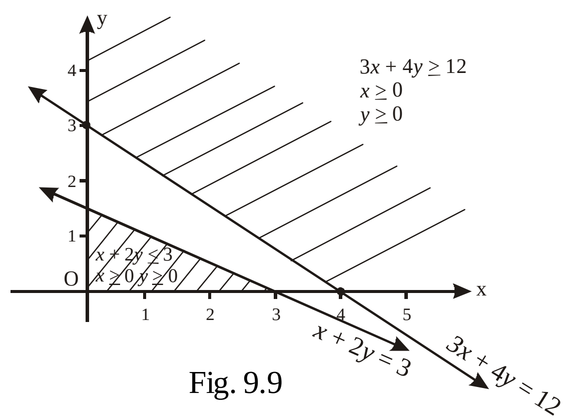


Fig. 9.9

Example 9.22 Solve the following system of linear inequations graphically :

$x - y < 2, \;\; 2x + y < 6; \; x \geq 0, \; y \geq 0.$

Solution : The given inequations are

$x - y < 2$ ... (i)
$2x + y < 6$ ... (ii)
$x \geq 0 \; ; \; y \geq 0$ ... (iii)

After representing the inequations $x - y < 2$, $2x + y < 6$, $x \geq 0$ and $y \geq 0$ on the graph we find the common region which is the bounded region $OABC$ as shown in Fig. 9.10

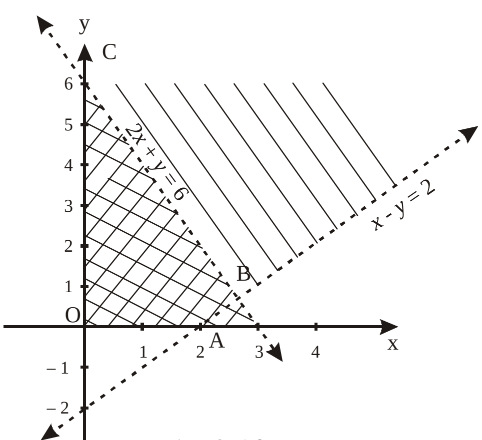


Fig. 9.10

CHECK YOUR PROGRESS 9.6

Solve each of the follwing systems of linear inequations in two variables graphically :

1. $x \geq 3,\ y \geq 1.$
2. $y \geq 2x, y \leq 2.$
3. $2x + y - 3 \geq 0, x - 2y + 1 \leq 0.$
4. $3x + 4y \leq 12,\ 4x+3y \leq 12,\ x \geq 0,\ y \geq 0$
5. $2x + 3y \geq 3,\ 3x + 4y \leq 18,\ 7x - 4y + 14 \geq 0,\ x - 6y \leq 3,\ x \geq 0, y \geq 0$
6. $x + y \geq 9,\ 3x + y \geq 12,\ x \geq 0,\ y \geq 0$
7. $x + y \geq 1;\ 2x + 3y \leq 6,\ x \geq 0,\ y \geq 0.$
8. $x + 3y \geq 10;\ x + 2y \leq 3,\ x - 2y \leq 2,\ x \geq 0\ ;\ y \geq 0$

LET US SUM UP

- Roots of the quadratic equation $ax^2 + bx + c = 0$ are complex and conjugate of each other, when $D < 0$. and $a, b, c \in R$.
- If α , β be the roots of the quadratic equation

  $ax^2 + bx + c = 0$ then $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$
- If α and β are the roots of a quadratic equation. then the equation is:

  $x^2 - (\alpha + \beta)x + \alpha\beta = 0$
- The maximum number of roots of an equation is equal to the degree of the equation.
  A statement involving a sign of inequality like, <, >, $\leq$, $\geq$, is called an inequation.
- The equation $ax + by + c = 0$ is a straight line which divides the xy-plane into two half planes which are represented by $ax + by + c \geq 0$ and $ax + by + c \leq 0$
- By the term, solution of a system of simultaneous linear inequalities we mean, finding all values of the ordered pairs (x,y) for which each linear inequalities of the system are satisfied.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=EoCeL4SPIcA
http://www.youtube.com/watch?v=FnrqBgot3jM
http://www.youtube.com/watch?v=-aTy1ED1m5I
http://www.youtube.com/watch?v=YBYu5aZPLeg
http://www.youtube.com/watch?v=2oGsLdAWxlk

TERMINAL EXERCISE

1. Show that the roots of the equation
$2(a^2+b^2)x^2+2(a+b)x+1=0$ are imaginary, when $a \neq b$

2. Show that the roots of the equation
$bx^2+(b-c)x = c+a-b$ are always real if those of
$ax^2+b(2x+1)=0$ are imaginary.

3. If α, β be the roots of the equation $2x^2-6x+5=0$, find the equation whose roots are:

(i) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (ii) $\alpha+\frac{1}{\beta}, \beta+\frac{1}{\alpha}$ (iii) $\alpha^2+\beta^2, \frac{1}{\alpha^2}+\frac{1}{\beta^2}$

Solve the following inequalities graphically.

4. $x \geq -2$
5. $y \leq 2.$
6. $x < 3$
7. $y \geq -3$
8. $5-3y \geq -4$
9. $2x-5 \leq 3.$
10. $3x-2y \leq 12$
11. $\frac{x}{3}+\frac{y}{5} \geq 1.$
12. $2x-3y \geq 0$
13. $x+2y \leq 0.$

Solve each of the following systems of linear inequalities in two variables graphically.

14. $-1 \leq x \leq 3,\ 1 \leq y \leq 4.$

15. $2x+3y \leq 6,\ \ 3x+2y \leq 6.$

16. $6x+5y \leq 150,$
$x+4y \leq 80$
$x \leq 15,\ x \geq 0,\ y \geq 0.$

17. $3x+2y \leq 24,\ x+2y \leq 16$
$x+y \leq 10,\ x \geq 0,\ y \geq 0$

18. $x+y \geq 3,\ 7x+6y \leq 42$
$x \leq 5, \quad y \leq 4$
$x \geq 0, \quad y \geq 0$

Solve that inequalities:

19. $\frac{3(x-2)}{5} \leq \frac{5(2-x)}{3}$

20. $37-(3x+5) \geq 9x-8(x-3)$

21. $\frac{(2x-1)}{3} \geq \frac{(3x-2)}{4}-\frac{(2-x)}{5}$

22. $5x+1 > -24, 5x-1 < 24$

23. $3x-7 > 2(x-6), 6-x > 11-2x$

24. $5(2x-7)-3(2x+3) \leq 0, 2x+19 \leq 6x+47.$

25. A solution of 8% boric acid is to be diluted by adding a 2% boric acid solution to it. The resulting mixture is to be move than 4% but less than 6% boric acid. If we have 640 litres of the 8% solution, how many litres of the 2% solution will have to be added?

26. How many litres of water will have to be added to 1125 litres of the 45% solution of acid so that the resulting mixture will contain more than 25% but less than 30% acid content?

ANSWERS

CHECK YOUR PROGRESS 9.1

1. (i) $-2\sqrt{3}, \dfrac{-4}{\sqrt{3}}$ (ii) $a-\sqrt{b},\ a+\sqrt{b}$ (iii) $-\dfrac{ab}{c}, \dfrac{c}{ab}$ (iv) $3\sqrt{2},\ \sqrt{2}$

CHECK YOUR PROGRESS 9.2

1. (i) $\dfrac{3\pm\sqrt{15}\,i}{4}$ (ii) $\dfrac{1\pm i}{\sqrt{2}}$ (iii) $\dfrac{\sqrt{5}\pm\sqrt{43}\,i}{8}$ (iv) $\dfrac{-\sqrt{2}\pm\sqrt{58}\,i}{6}$

2. $-1, \dfrac{1}{2}$

CHECK YOUR PROGRESS 9.3

1. (i) $\dfrac{b^2-2ac}{c^2}$ (ii) $\dfrac{(b^2-2ac)^2-2a^2c^2}{c^4}$

2. (i) $25x^2-6x+9=0$ (ii) $625x^2-90x+81=0$

4. $q^2-5p^2=0$

CHECK YOUR PROGRESS 9.4

1. $\dfrac{-1\pm\sqrt{7}i}{2}$

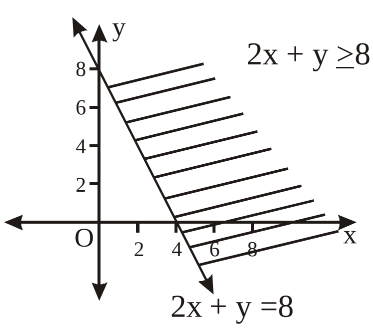
2. $\dfrac{\sqrt{2}\pm\sqrt{34}i}{2}$

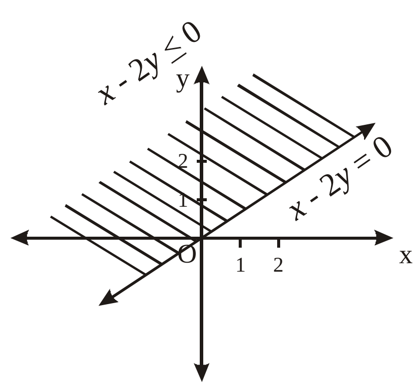
3. $\dfrac{-1\pm\sqrt{7}i}{2\sqrt{2}}$

4. $\dfrac{-1\pm\sqrt{19}i}{2}$

5. $\dfrac{-3\pm\sqrt{11}i}{2}$

CHECK YOUR PROGRESS 9.5

1. 

2. 

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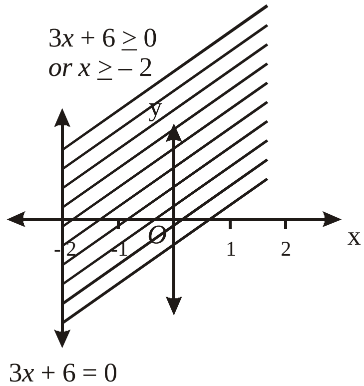
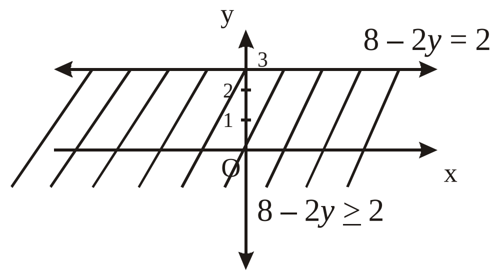
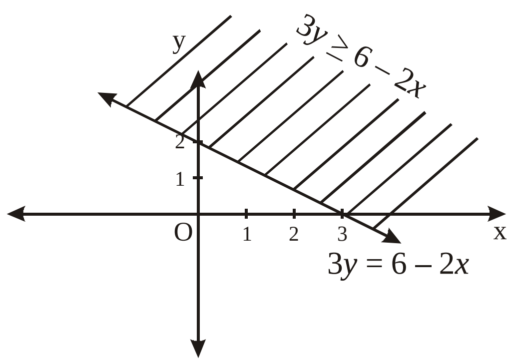
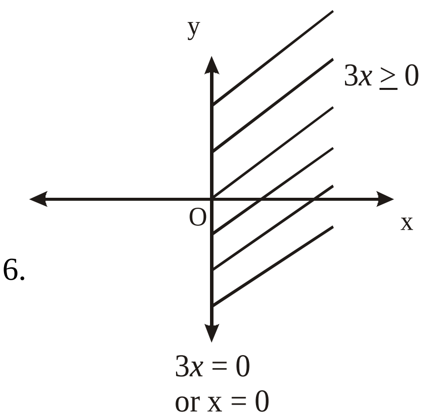
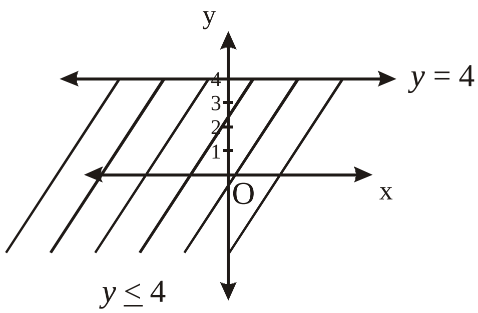
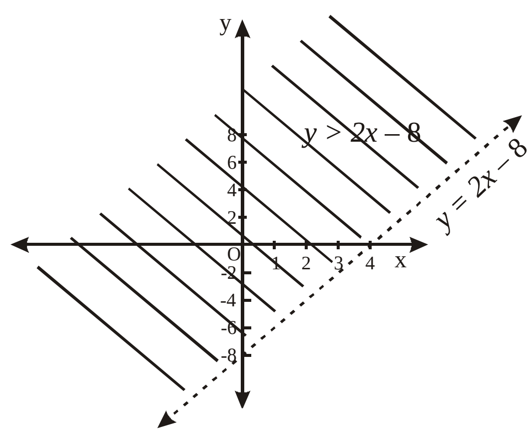

3x + 6 ≥ 0
or x ≥ – 2
y
O
-2
-1
1
2
x
3x + 6 = 0

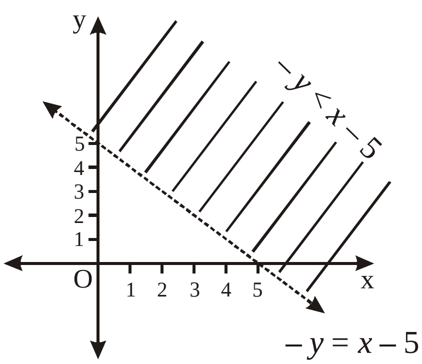
4.

y
8 – 2y = 2
3
2
1
O
x
8 – 2y ≥ 2

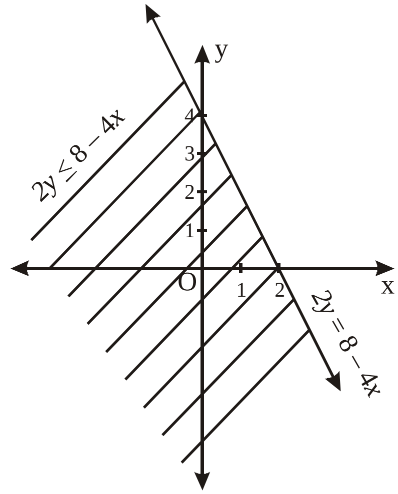
5.

y
3y ≥ 6 – 2x
2
1
O
1
2
3
x
3y = 6 – 2x

6.

y
3x ≥ 0
O
x
3x = 0
or x = 0

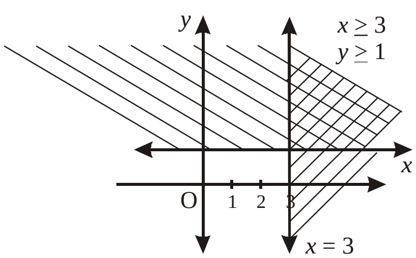
7.

y
y = 4
4
3
2
1
O
x
y ≤ 4

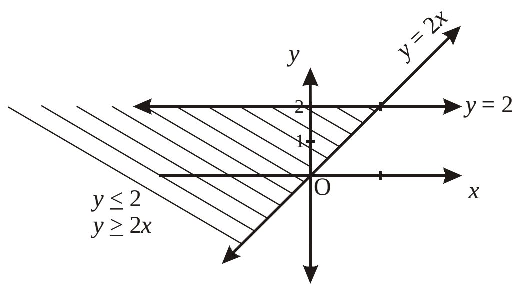
8.

y
8
6
4
2
O
-2
-4
-6
-8
1
2
3
4
x
y > 2x – 8
y = 2x – 8

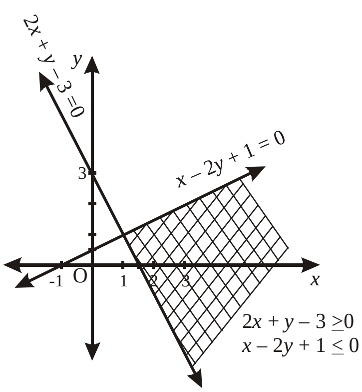
9. 

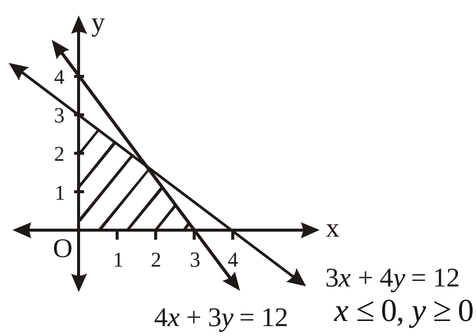
10. 

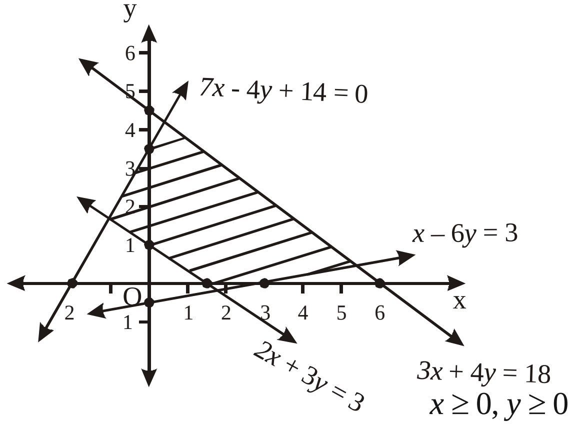
CHECK YOUR PROGRESS 9.6

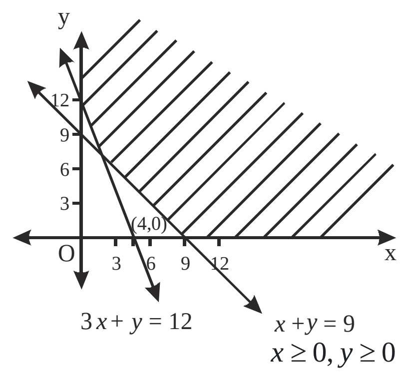
1. 

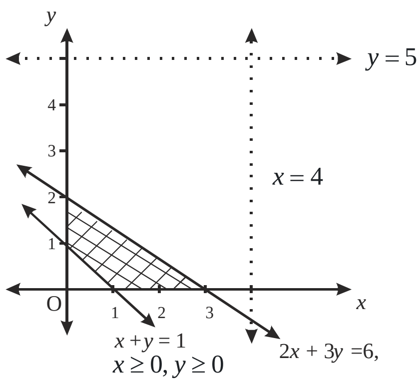
2. 

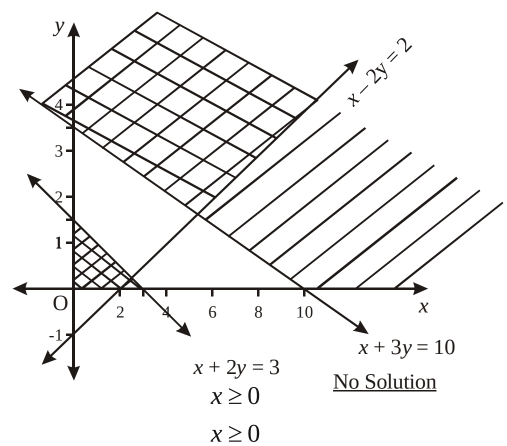
(3) 

(4) 

5. 

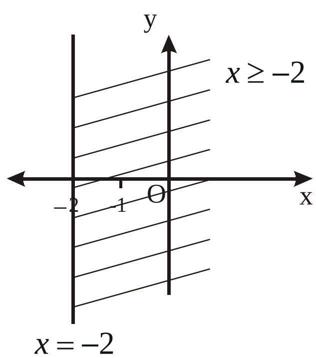
6. 

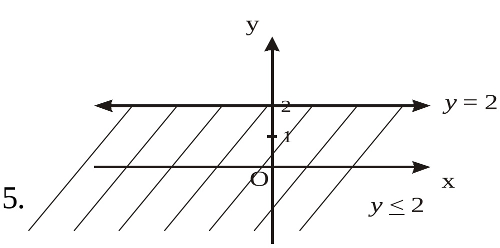
7 . 

8. 

TERMINAL EXERCISE

3. (i) $5x^2 - 8x + 5 = 0$ (ii) $10x^2 - 42x + 49 = 0$ (iii) $25x^2 - 116x + 64 = 0$

4. 

5. 

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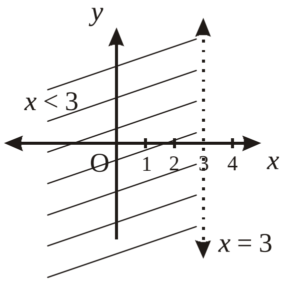

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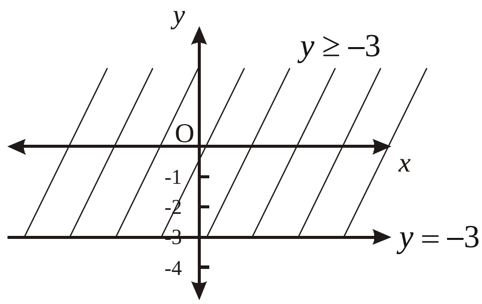

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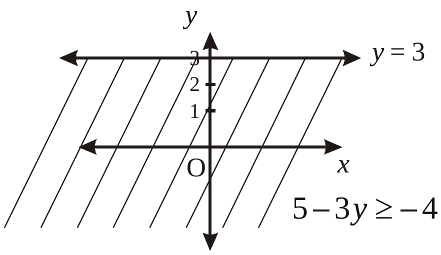

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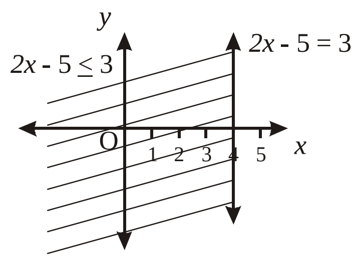

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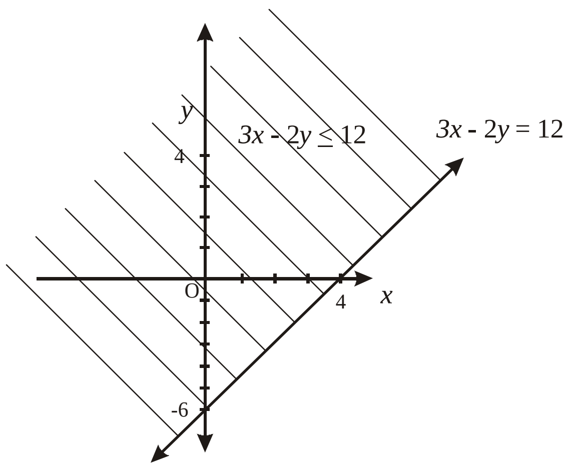

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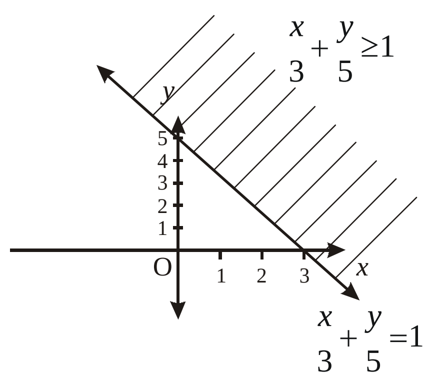

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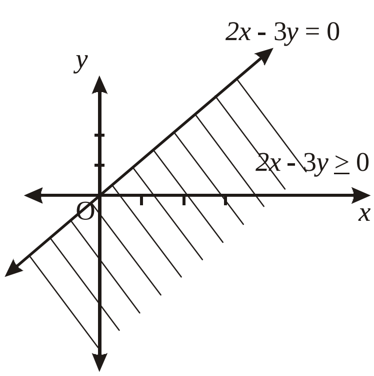
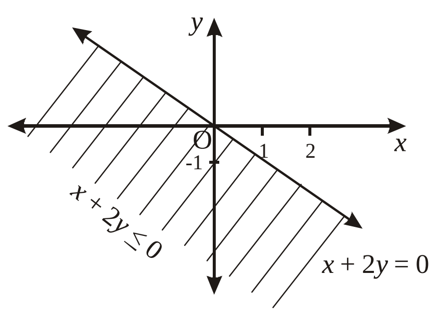
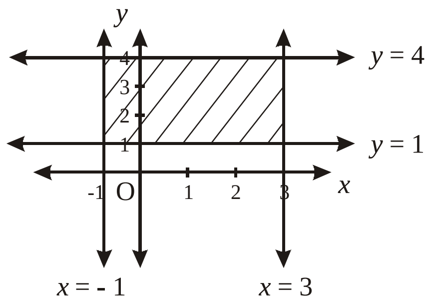
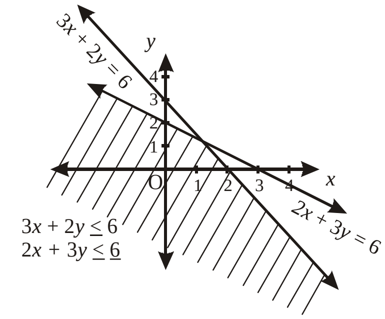
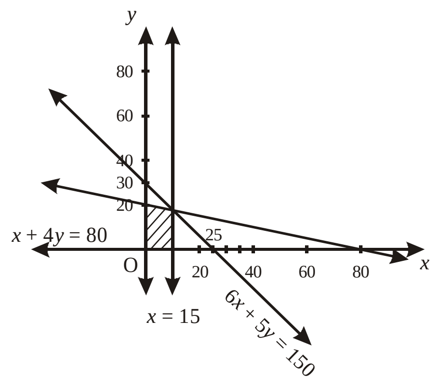


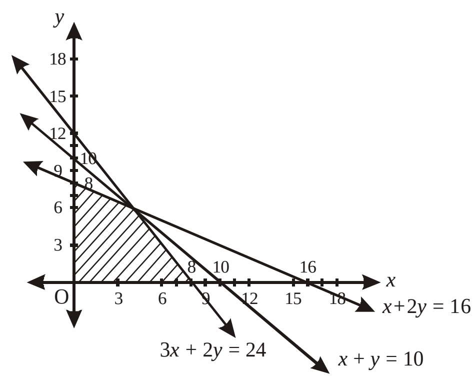
13.

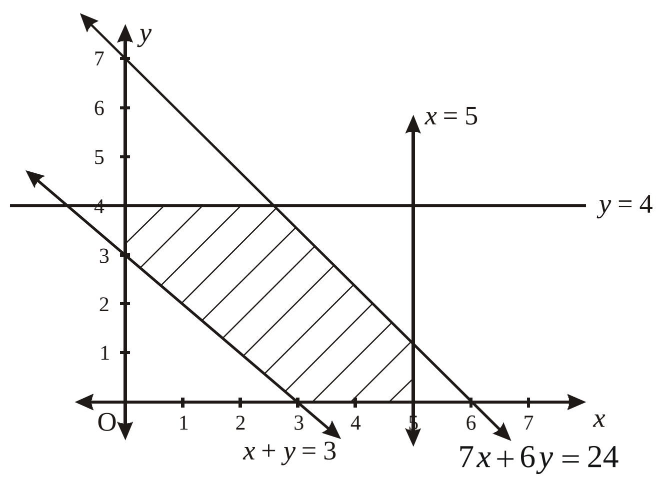


11. 

12. 

13. 

14. 

15. 

19. $(-\infty, 2]$ 20. $(-\infty, 2]$

21. $(-\infty, 2]$ 22. $(-5, 5)$ 23. $(5, \infty)$ 24. $[-7, 11]$

25. More then 320 litre but less then 1280 litres

26. More then 562.5 litres but less then 900 litres